Course 32B/2

Fall 2020

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Student ID:

Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Pr 7	Pr 8	Pr 9	Total
$\overline{10}$	$\overline{14}$	$\overline{6}$	$\overline{10}$	$\overline{18}$	$\overline{10}$	$\overline{12}$	$\overline{10}$	$\overline{10}$	$\overline{100}$

Midterm 2

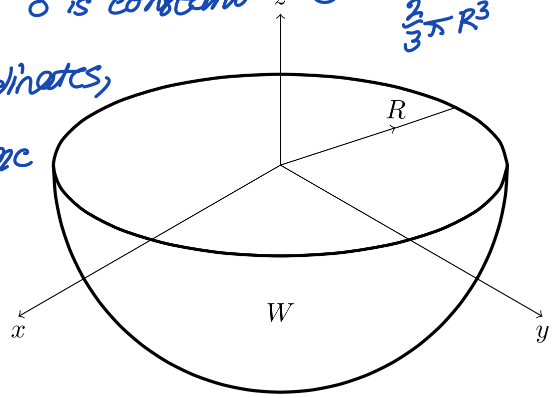
Please print your name and student ID in the designated space at the top of the page. Show your work! Answers unsupported by work yield no credit.

In Problems 1–3, you will be asked to find various characteristics of a homogeneous lower hemi-sphere W of radius R and mass M centered at the origin, its equatorial plane coinciding with the (xy)-plane as shown on the picture below.

To find the centroid, assume δ is constant $\equiv c = \dfrac{M}{\frac{2}{3}\pi R^3} = \dfrac{3M}{2\pi R^3}$

Moreover, in polar coordinates, we have our bounds for the surface as

$\rho: 0 \to R$
$\phi: \frac{\pi}{2} \to \pi$
$\theta: 0 \to 2\pi$



So now we can calculate

Problem 1 Find the centroid of W. **10 pts**

$$M_1 = \iiint x\,\delta(x,y,z)\,dV = \int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{R} c\,\rho\sin\phi\cos\theta \cdot \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi} c\,\rho^3\sin^2\phi\cos\theta\,d\theta\,d\rho\,d\phi$$

[as θ is not dependant on the other integrals]

$$= \int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{R} c\,\rho^3\sin^3\phi\,\left[\sin\theta\right]_0^{2\pi} = 0$$

$$M_2 = \iiint y\,\delta(x,y,z)\,dV = \int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{R} c\,\rho\sin\phi\sin\theta\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi} c\,\rho^3\sin^2\phi\sin\theta\,d\theta\,d\rho\,d\phi = \int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{R} c\,\rho^3\sin^2\phi\,\left[-\cos\theta\right]_0^{2\pi}\,d\rho\,d\phi = 0$$

$$M_3 = \iiint z\,\delta(x,y,z)\,dV = \int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{\pi} c\,\rho\cos\phi\;\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= c\int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi}\frac{\rho^4}{4}\cos\phi\sin\phi\,\Big|_{\rho=0}^{R}\,d\phi\,d\theta = \frac{R^4c}{4}\int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi}\sin\phi\cos\phi\,d\phi\,d\theta$$

$$= \frac{R^4 c}{8} \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{2}}^{\pi} \sin 2\phi \, d\phi \, d\theta = \frac{R^4 c}{8} \int_{\theta=0}^{2\pi} \frac{-\cos 2\phi}{2} \Big|_{\phi=\frac{\pi}{2}}^{\pi} d\theta = \frac{R^4 c}{8} \int_{\theta=0}^{2\pi} -\frac{1}{2} - \frac{1}{2} \, d\theta$$

$$= \frac{-R^4 c}{8} \cdot 2\pi = \frac{-2\pi R^4 c}{8} = -\frac{\pi R^4}{4} \cdot \frac{3m}{2\pi R^3} = \frac{-3mR}{8}$$

So the centroid is at $\left(\frac{0}{m}, \frac{0}{m}, \frac{-3mR}{8m}\right) = \left(0, 0, \frac{-3R}{8}\right)$

Problem 2 Find the moments of inertia of W with respect to the x, y, and z axes. Hint: it will help a lot if you think about the symmetries of the sphere! **14 pts**

- I_x **6 pts**

$$I_x = \iiint_W (y^2 + z^2)\, \delta(x,y,z)\, dV = \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{2}}^{\pi} \int_{\rho=0}^{R} (\rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi)\, c \cdot \rho^2 \sin\phi \, dV$$

$$= c \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{2}}^{\pi} \int_{\rho=0}^{R} \rho^4 \sin^3\phi \sin^2\theta \, d\rho\, d\phi\, d\theta + c \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{2}}^{\pi} \int_{\rho=0}^{R} \rho^4 \sin\phi \cos^2\phi \, d\rho\, d\phi\, d\theta$$

$$= c \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{2}}^{\pi} \frac{R^5}{5} \sin^3\phi \sin^2\theta \, d\phi\, d\theta + c \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{2}}^{\pi} \frac{R^5}{5} \sin\phi \cos^2\phi \, d\phi\, d\theta$$

$$= \frac{R^5 c}{5} \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{2}}^{\pi} \sin\phi(1-\cos^2\phi) \sin^2\theta \, d\phi\, d\theta + \frac{R^5 c}{5} \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{2}}^{\pi} \sin\phi \cos^2\phi \, d\phi\, d\theta$$

Let $\cos\phi = u$

$-du = \sin\phi \, d\phi$

$\phi = \frac{\pi}{2} \to u = 0, \ \phi = \pi, \ u = -1$

Let $\cos\phi = v$

$-dv = \sin\phi \, dv$

$\phi = \frac{\pi}{2} \to v = 0, \ \phi = \pi, \ v = -1$

$$= \frac{R^5 c}{5} \int_{\theta=0}^{2\pi} \int_{u=0}^{-1} (u^2 - 1) \sin^2\theta \, du\, d\theta + \frac{R^5 c}{5} \int_{\theta=0}^{2\pi} \int_{v=0}^{-1} -v^2 \, dv\, d\theta$$

$$= \frac{R^5 c}{5} \int_{\theta=0}^{2\pi} \left(\frac{u^3}{3} - u\right)_0^{-1} \sin^2\theta \, d\theta + \frac{R^5 c}{5} \int_{\theta=0}^{2\pi} \frac{-v^3}{3}\Big|_0^{-1} d\theta$$

$$= \frac{R^5 c}{5} \int_{\theta=0}^{2\pi} \left(\frac{-1}{3} + 1\right) \sin^2\theta \, d\theta + \frac{R^5 c}{5} \int_{\theta=0}^{2\pi} \frac{1}{3} \, d\theta$$

$$= \frac{2R^5 c}{15} \int_{\theta=0}^{2\pi} \sin^2\theta \, d\theta + \frac{R^5 c}{15} \int_{\theta=0}^{2\pi} d\theta$$

$$= \frac{2R^5 c}{15} \int_{\theta=0}^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta + \frac{2\pi R^5 c}{15}$$

The problem continues to the next page.

$$= \frac{2R^5 c}{15} \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_{\theta=0}^{2\pi} + \frac{2\pi R^5 c}{15} = \frac{2R^5 c}{15} (\pi) + \frac{2\pi R^5 c}{15} = \frac{4\pi R^5 c}{15}$$

3

$$= \frac{4\pi R^5}{15} \times \frac{3m}{2\pi R^3} = \frac{2}{5} mR^2$$

The hemisphere is symmetrical in the xy plane along any axis in the xy plane. This means that the moment of inertia for x axis will be the same for y.

- I_y $= \frac{2}{5} mR^2 = I_x$ **2 pts**

- I_z **6 pts**

$$I_z = \iiint_W (x^2+y^2)\,\delta(x,y,z)\,dV = c\int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{R} (\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= c\int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{R} \rho^2\sin^2\phi(\sin^2\theta+\cos^2\theta)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = c\int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{R} \rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta$$

$$= c\int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi} \frac{\rho^5}{5}\Big|_{\rho=0}^{R} \sin^3\phi\, d\phi\, d\theta = c\int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi} \frac{R^5}{5}\sin^3\phi\, d\phi\, d\theta$$

$$= \frac{cR^5}{5}\int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{2}}^{\pi} \sin\phi(1-\cos^2\phi)\, d\phi\, d\theta$$

$t = \cos\phi$, $dt = -\sin\phi\, d\phi$

$\phi = \frac{\pi}{2} \to t = 0$, $\phi = \pi \to t = -1$

$$= \frac{cR^5}{5}\int_{\theta=0}^{2\pi}\int_{t=0}^{-1} (t^2-1)\, dt\, d\theta$$

$$= \frac{cR^5}{5}\int_{\theta=0}^{2\pi}\left(\frac{t^3}{3}-t\right)_0^{-1} d\theta = \frac{cR^5}{5}\int_{\theta=0}^{2\pi}\left(-\frac{1}{3}+1\right) d\theta = \frac{2cR^5}{15}\cdot 2\pi$$

$$= \frac{4\pi cR^5}{15} = \frac{4\pi R^5}{15}\cdot\frac{3m}{2\pi R^3} = \frac{2}{5} mR^2$$

Problem 3 Find the radii of gyration of W with respect to the x, y, and z axes. **6 pts**

- $R_x = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\frac{2}{5}mR^2}{m}} = \sqrt{\frac{2}{5}}R$ **2 pts**

- $R_y = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{\frac{2}{5}mR^2}{m}} = \sqrt{\frac{2}{5}}R$ **2 pts**

- $R_z \sqrt{\frac{I_z}{m}} = \sqrt{\frac{\frac{2}{5}mR^2}{m}} = \sqrt{\frac{2}{5}}R$ **2 pts**

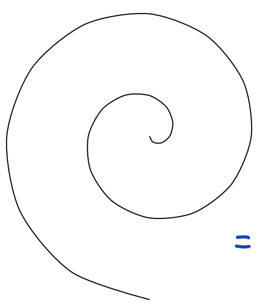
Problem 4 Find the length of a *circle involute* **10 pts**

$$x = R(\cos t + t\sin t), \qquad y = R(\sin t - t\cos t), \qquad 0 \le t \le 2.$$

We have $\vec{r}(t) = \langle R(\cos t + t\sin t),\ R(\sin t - t\cos t)\rangle$

Then $\vec{r}\,'(t) = \langle R(-\sin t + t\cos t + \sin t), R(\cos t - \cos t + t\sin t)\rangle = \langle Rt\cos t, Rt\sin t\rangle$

Then $|\vec{r}\,'(t)| = \sqrt{R^2t^2\cos^2 t + R^2t^2\sin^2 t} = Rt$



We want to find $\int_e 1 \cdot ds$

$= \int_{t=0}^{2} Rt\,dt = \frac{Rt^2}{2}\Big|_{t=0}^{2} = 2R$

We know that magnitude of Force $= \frac{kQq}{r^2} = \frac{kQq}{(x^2+y^2+z^2)}$ (in 3 dimensions)

Problem 5 **18 pts**

- Use vector notations to formulate Coulomb's Law for the electric point-charges Q and q. **2 pts**

A unit vector along the direction of force from a point charge is $\vec{U} = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$

Then the force emerging from a single point charge will be $= \frac{kQq}{x^2+y^2+z^2}\vec{U} = \left\langle \frac{kQqx}{(x^2+y^2+z^2)^{3/2}}, \frac{kQqy}{(x^2+y^2+z^2)^{3/2}}, \frac{kQqz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$

Then field will just be the force/(test charge q)

$$\vec{F} = \left\langle \frac{kQx}{(x^2+y^2+z^2)^{3/2}}, \frac{kQy}{(x^2+y^2+z^2)^{3/2}}, \frac{kQz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

- Find the potential U an electric charge Q placed at the origin creates at a point $P = (x, y, z)$. Check that the found U is the potential by computing ∇U. **6 pts**

For a point P, the distance from the origin is $\sqrt{x^2+y^2+z^2}$.

Then the potential will be given by $U = \frac{-kQ}{\sqrt{(x^2+y^2+z^2)}}$

We can verify this.

$$\nabla U = \left\langle \frac{\partial}{\partial x}\left(\frac{-kQ}{\sqrt{(x^2+y^2+z^2)}}\right), \frac{\partial}{\partial y}\left(\frac{-kQ}{\sqrt{(x^2+y^2+z^2)}}\right), \frac{\partial}{\partial z}\left(\frac{-kQ}{\sqrt{(x^2+y^2+z^2)}}\right) \right\rangle$$

$$= \left\langle -\frac{1}{2}\cdot\frac{-kQ\cdot 2x}{(x^2+y^2+z^2)^{3/2}}, -\frac{1}{2}\cdot\frac{-kQ\cdot 2y}{(x^2+y^2+z^2)^{3/2}}, -\frac{1}{2}\cdot\frac{-kQ\cdot 2z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$= \left\langle \frac{kQx}{(x^2+y^2+z^2)^{3/2}}, \frac{kQy}{(x^2+y^2+z^2)^{3/2}}, \frac{kQz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

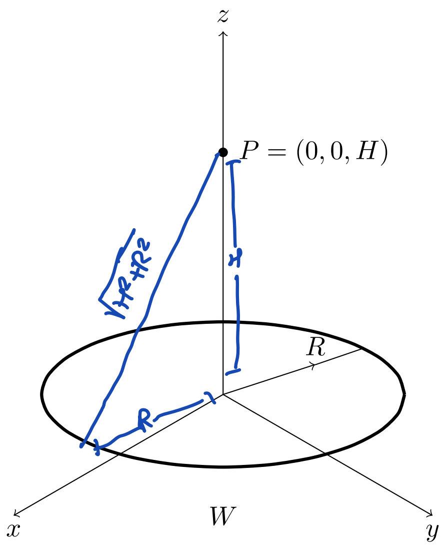
$$= \vec{F}.$$

Thus U was the potential of $\vec{F}$.

The problem continues to the next page.

- An electric charge Q is evenly distributed along a thin circular wire of radius R located in the xy-plane and centered at the origin as on the picture below. **10 pts**

Find the potential of the wire at the point $P = (0, 0, H)$.



We can say that the linear charge density along the wire is $\lambda = c = \frac{\text{charge}}{\text{length}}$

$$\lambda = \frac{Q}{2\pi R}.$$

We know that the potential can be calculated by $k\int_C \frac{f(x,y,z)}{\sqrt{H^2+R^2}}\,ds.$

here f is same as λ

C we know is given by the parametrization $\vec{r}(t) = \langle R\cos t, R\sin t, 0\rangle$

$\vec{r}\,'(t) = \langle -R\sin t, R\cos t, 0\rangle$, $\|\vec{r}\,'(t)\| = \sqrt{R^2\sin^2 t + R^2\cos^2 t} = R$

Then $ds = \|\vec{r}\,'(t)\|\,dt = R\,dt$. Note since we are going full circle, $t: 0 \to 2\pi$

Then potential will be given by $k\int_0^{2\pi} \frac{\lambda \cdot R\,dt}{\sqrt{H^2+R^2}} = \frac{2\pi k R\lambda}{\sqrt{H^2+R^2}} = \frac{2\pi k R}{\sqrt{H^2+R^2}} \times \frac{Q}{2\pi R}$

$$= \frac{kQ}{\sqrt{H^2+R^2}}$$

Problem 6 **10 pts**

Find the area of the region D in the first quadrant bounded by the curves on the picture below. Leave the answer in the symbolic form, but simplify it as much as possible.

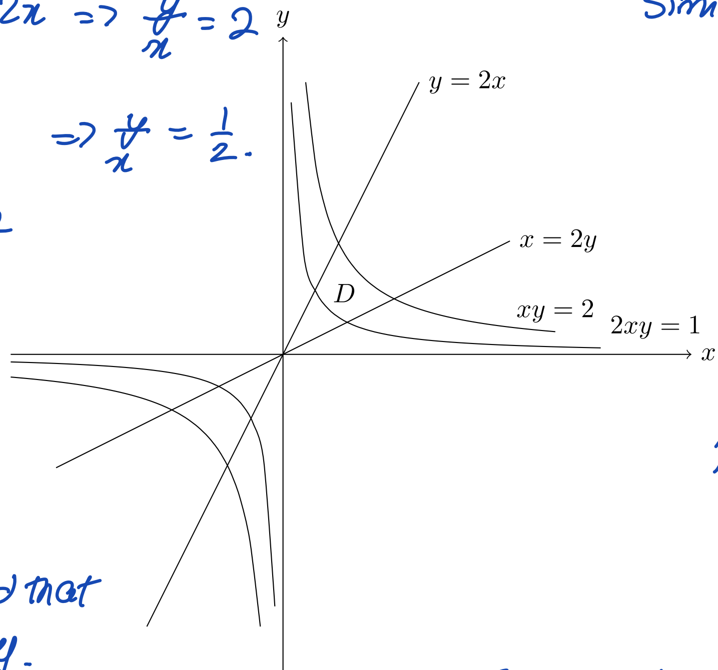
we have $y = 2x \Rightarrow \frac{y}{x} = 2$

and $x = 2y \Rightarrow \frac{y}{x} = \frac{1}{2}$.

So $\frac{1}{2} \le \frac{y}{x} \le 2$

Similarly $xy = 2$ and $2xy = 1$

So $\frac{1}{2} \le xy \le 2$.



Let $u = \frac{y}{x}$

Then $\frac{1}{2} \le u \le 2$

Let $v = xy$

Then $\frac{1}{2} \le v \le 2$.

We have defined that $u = \frac{y}{x}$, $v = xy$. $\Rightarrow y = ux$, $v = x^2 u \Rightarrow x = \sqrt{\frac{u}{v}}$ [+ve roots as we are in the first coordinate]

Similarly $x = \frac{y}{u}$, $\Rightarrow v = \frac{y^2}{u} \Rightarrow y = \sqrt{vu}$

Then $G(u,v) = \left(\sqrt{\frac{u}{v}}, \sqrt{uv} \right)$.

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & \frac{-\sqrt{u}}{2v^{3/2}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix} = \frac{1}{4} \cdot \frac{1}{v} + \frac{1}{4} \cdot \frac{1}{v} = \frac{1}{2v}$$

The domain D in x-y coordinates will translate to Box $B = [\frac{1}{2}, 2] \times [\frac{1}{2}, 2]$ in u-v coordinates.

So we want to find $\int_{u=\frac{1}{2}}^{2} \int_{v=\frac{1}{2}}^{2} 1 \cdot \frac{1}{2v} \, dv \, du = \int_{u=\frac{1}{2}}^{2} \frac{1}{2} \ln|v| \Big|_{v=\frac{1}{2}}^{2} du$

$= \frac{1}{2} \int_{u=\frac{1}{2}}^{2} \ln\left|\frac{2}{\frac{1}{2}}\right| du = \ln 2 \int_{u=\frac{1}{2}}^{2} du = \ln 2 \, (u)_{\frac{1}{2}}^{2} = \frac{3}{2} \ln 2$

Problem 7 **12 pts**

- Write down the formulas that express the rectangular coordinates x, y, and z as functions of the spherical coordinates ρ, ϕ, and θ. **2 pts**

$x(\rho,\phi,\theta) = \rho\sin\phi\cos\theta$

$y(\rho,\phi,\theta) = \rho\sin\phi\sin\theta$

$z(\rho,\phi,\theta) = \rho\cos\phi$

- Find the Jacobian of the coordinate change. **10 pts**

We have $G(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

$$Jac(G) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2 \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \sin\phi\cos\theta \\ \cos\phi & -\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{vmatrix}$$

$$= \frac{\rho^2}{\cos\phi} \begin{vmatrix} \sin\phi\cos\phi\cos\theta & \sin\phi\cos\phi\cos\theta & -\sin\theta \\ \sin\phi\cos\phi\sin\theta & \sin\phi\cos\phi\sin\theta & \cos\theta \\ \cos^2\phi & -\sin^2\phi & 0 \end{vmatrix}$$ [~~Subtract~~ col2 to col1]

$$= \frac{\rho^2}{\cos\phi} \begin{vmatrix} 0 & \sin\phi\cos\phi\cos\theta & -\sin\theta \\ 0 & \sin\phi\cos\phi\sin\theta & \cos\theta \\ 1 & -\sin^2\phi & 0 \end{vmatrix} = \frac{\rho^2\sin\phi}{\cos\phi} \begin{vmatrix} 0 & \cos\phi\cos\theta & -\sin\theta \\ 0 & \cos\phi\sin\theta & \cos\theta \\ 1 & -\sin\phi & 0 \end{vmatrix}$$

$$= \frac{\rho^2\sin\phi}{\cos\phi}\left(1\left(\cos\phi\cos^2\theta + \cos\phi\sin^2\theta\right)\right) = \frac{\rho^2\sin\phi}{\cos\phi}\cos\phi \cdot (\cos^2\theta + \sin^2\theta)$$

$$= \rho^2\sin\phi$$

Problem 8 **10 pts**

Find the volume of the following ellipsoid. Hint: a coordinate change helps!

$$\left.\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right\} \rightarrow D$$

Let us switch to coordinates such that $p = \frac{x}{a}$, $q = \frac{y}{b}$, $r = \frac{z}{c}$.

Then $G(p,q,r) = (ap, bq, cr)$. $Jac(G) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$.

We want to now find the volume of $p^2+q^2+r^2=1$

However, we know that this is just a unit sphere! Call this sphere in new coordinate system as ω.

So, the volume of $\iiint_D dV = \iiint_\omega abc\, dV = abc \iiint_\omega dV$

$$= abc \cdot \frac{4\pi}{3} = \frac{4\pi abc}{3}.$$

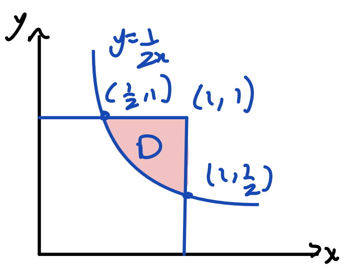
Problem 9 **10 pts**

The real numbers X and Y are randomly and independently chosen between zero and one. The joint probability density is

$$p(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1] \times [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability P that the product XY is at least $1/2$.

For $xy \geq \frac{1}{2} \Rightarrow y \geq \frac{1}{2x}$. The domain will then look like



So we can integrate to find the probability over the domain as follows

$$\int_{x=\frac{1}{2}}^{1}\int_{y=\frac{1}{2x}}^{1} p(x,y)\,dy\,dx$$

$$= \int_{x=\frac{1}{2}}^{1}\int_{y=\frac{1}{2x}}^{1} 1 \cdot dy\,dx = \int_{x=\frac{1}{2}}^{1} y\Big|_{y=\frac{1}{2x}}^{1} dx$$

$$= \int_{x=\frac{1}{2}}^{1} 1-\frac{1}{2x}\,dx = \left[x - \frac{1}{2}\ln|x|\right]_{\frac{1}{2}}^{1}$$

$$= \left(1-0-\frac{1}{2}+\frac{1}{2}\ln\left|\frac{1}{2}\right|\right)$$

$$= \frac{1}{2} - \frac{1}{2}\ln 2 \approx 0.15$$