Course 32B/2

UCLA Department of Mathematics

Fall 2020

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Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Pr 7	Pr 8	Pr 9	Total
$\overline{10}$	$\overline{14}$	$\overline{6}$	$\overline{10}$	$\overline{18}$	$\overline{10}$	$\overline{12}$	$\overline{10}$	$\overline{10}$	$\overline{100}$

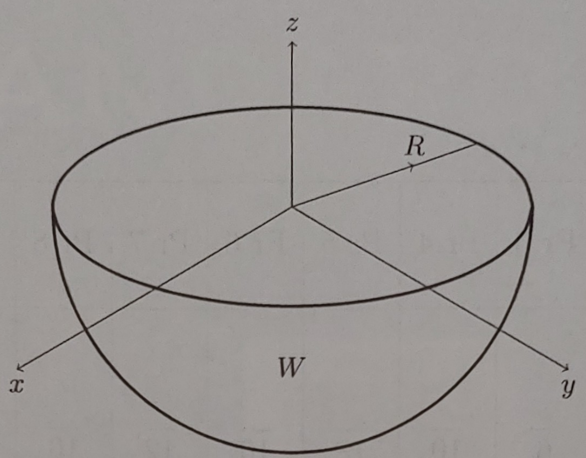
Midterm 2

Please print your name and student ID in the designated space at the top of the page. Show your work! Answers unsupported by work yield no credit.

In Problems 1–3, you will be asked to find various characteristics of a homogeneous lower hemi-sphere W of radius R and mass M centered at the origin, its equatorial plane coinciding with the (xy)-plane as shown on the picture below.

$\mu(x,y,z) \equiv \text{const}$

$\mu = \frac{M}{V} = \frac{M}{\frac{2}{3}\pi R^3} = \frac{3M}{2\pi R^3}$



Problem 1 Find the centroid of W. **10 pts**

By symmetry, $\bar{x} = \bar{y} = 0$.

$M = \iiint_W \mu(x,y,z)\, dV = \mu \iiint_W dV = \frac{1}{2}\mu\left(\frac{4}{3}\pi R^3\right) = \frac{2}{3}\mu\pi R^3$

$\bar{z} = \frac{3}{2\mu\pi R^3}\iiint_W z\mu(x,y,z)\, dV = \frac{3}{2\pi R^3}\int_{\pi/2}^{\pi}\int_0^{2\pi}\int_0^R (\rho\cos\phi)\rho^2\sin\phi \, d\rho\, d\theta\, d\phi$

$= \frac{3}{2\pi R^3}\int_0^{2\pi} d\theta \int_{\pi/2}^{\pi}\int_0^R \rho^3 \sin\phi\cos\phi \, d\rho\, d\phi = \frac{3}{R^3}\int_{\pi/2}^{\pi} \sin\phi\cos\phi \left(\frac{1}{4}\rho^4\right)\Big|_0^R d\phi$

Let $w = \sin\phi$
$dw = \cos\phi\, d\phi$

$= \frac{3}{4}R\int_{\pi/2}^{\pi} \sin\phi\cos\phi\, d\phi = \frac{3}{4}R\int_1^0 w\, dw = -\frac{3}{4}R\left(\frac{1}{2}w^2\right)\Big|_0^1 = -\frac{3}{8}R$

$\boxed{C = (0, 0, -\frac{3}{8}R)}$

$\mu = \frac{3M}{2\pi R^3}$

Problem 2 Find the moments of inertia of W with respect to the x, y, and z axes. Hint: it will help a lot if you think about the symmetries of the sphere! **14 pts**

- I_x **6 pts**

$$I_x = \iiint_W (y^2+z^2)\,\mu\, dV = \mu \int_{\pi/2}^{\pi}\int_0^{2\pi}\int_0^R \left[(\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2\right]\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \mu \int_{\pi/2}^{\pi}\int_0^{2\pi}\int_0^R \rho^4\sin^3\phi\sin^2\theta + \rho^4\sin\phi\cos^2\phi\, d\rho\, d\theta\, d\phi = \mu\int_{\pi/2}^{\pi}\int_0^{2\pi} \sin^3\phi\sin^2\theta\left(\frac{1}{5}\rho^5\right)\Big|_0^R + \sin\phi\cos^2\phi\left(\frac{1}{5}\rho^5\right)\Big|_0^R\, d\theta\, d\phi$$

$$= \frac{1}{5}\mu R^5 \int_{\pi/2}^{\pi}\int_0^{2\pi} \sin^3\phi\left(\frac{1-\cos 2\theta}{2}\right) + \sin\phi\cos^2\phi\, d\theta\, d\phi = \frac{1}{5}\mu R^5\int_{\pi/2}^{\pi} \sin^3\phi\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} + \sin\phi\cos^2\phi\,(\theta)\Big|_0^{2\pi}\, d\phi$$

$$= \frac{1}{5}\mu R^5\int_{\pi/2}^{\pi} \pi\sin^3\phi + 2\pi\sin\phi\cos^2\phi\, d\phi = \frac{\pi}{5}\mu R^5\int_{\pi/2}^{\pi}(1-\cos^2\phi)\sin\phi + 2\sin\phi\cos^2\phi\, d\phi$$

Let $w = \cos\phi$

$dw = -\sin\phi\, d\phi$

$$= -\frac{\pi}{5}\mu R^5\int_0^{-1} 1 - w^2 + 2w^2\, dw = \frac{\pi}{5}\mu R^5\int_{-1}^{0} 1 + w^2\, dw = \frac{\pi}{5}\mu R^5\left(w + \frac{1}{3}w^3\right)\Big|_{-1}^{0} = \frac{\pi}{5}\mu R^5\left(1+\frac{1}{3}\right) = \frac{4\pi}{15}\mu R^5$$

$$= \frac{4\pi}{15}R^5\left(\frac{3M}{2\pi R^3}\right) = \boxed{\frac{2}{5}MR^2}$$

The problem continues to the next page.

$\mu = \frac{3M}{2\pi R^3}$

- I_y **2 pts**

By symmetry, $I_x = I_y$. (The distribution of mass relative to the x-axis is the same as its distribution relative to the y-axis.)

$\boxed{I_y = \frac{2}{5}MR^2}$

- I_z **6 pts**

$$I_z = \iiint_W (x^2+y^2)\mu\, dV = \mu\int_{\pi/2}^{\pi}\int_0^{2\pi}\int_0^R \left[(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2\right]\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \mu\int_{\pi/2}^{\pi}\int_0^{2\pi}\int_0^R \rho^4\sin^3\phi\, d\rho\, d\theta\, d\phi = \mu\int_{\pi/2}^{\pi}\int_0^{2\pi}\sin^3\phi\left(\frac{1}{5}\rho^5\right)\Big|_0^R d\theta\, d\phi$$

$$= \frac{1}{5}\mu R^5\int_{\pi/2}^{\pi}\int_0^{2\pi}\sin^3\phi\, d\theta\, d\phi = \frac{1}{5}\mu R^5\int_{\pi/2}^{\pi}\sin^3\phi(\theta)\Big|_0^{2\pi} d\phi$$

Let $w = \cos\phi$

$dw = -\sin\phi\, d\phi$

$$= \frac{2\pi}{5}\mu R^5\int_{\pi/2}^{\pi}(1-\cos^2\phi)\sin\phi\, d\phi = -\frac{2\pi}{5}\mu R^5\int_0^{-1} 1-w^2\, dw$$

$$= \frac{2\pi}{5}\mu R^5\left(w-\frac{1}{3}w^3\right)\Big|_{-1}^{0} = \frac{2\pi}{5}\mu R^5\left(1-\frac{1}{3}\right) = \frac{4\pi}{15}\mu R^5$$

$$= \frac{4\pi}{15}R^5\left(\frac{3M}{2\pi R^3}\right) = \boxed{\frac{2}{5}MR^2}$$

Problem 3 Find the radii of gyration of W with respect to the x, y, and z axes. **6 pts**

- R_x $\left(r_g = \sqrt{\frac{I}{M}}\right)$ **2 pts**

  $R_x = \sqrt{\frac{2}{5}MR^2 \cdot \frac{1}{M}}$

  $\boxed{R_x = \frac{\sqrt{10}}{5}R}$

- R_y **2 pts**

  $R_y = \sqrt{\frac{2}{5}MR^2 \cdot \frac{1}{M}}$

  $\boxed{R_y = \frac{\sqrt{10}}{5}R}$

- R_z **2 pts**

  $R_z = \sqrt{\frac{2}{5}MR^2 \cdot \frac{1}{M}}$

  $\boxed{R_z = \frac{\sqrt{10}}{5}R}$

Problem 4 Find the length of a *circle involute* **10 pts**

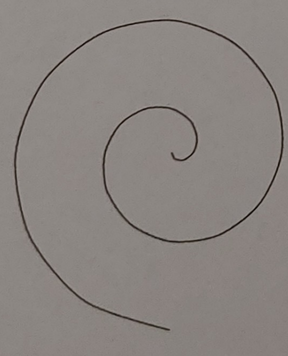
$$x = R(\cos t + t\sin t), \qquad y = R(\sin t - t\cos t), \qquad 0 \le t \le 2.$$

$\dot{x} = -R\sin t + Rt\cos t + R\sin t$

$\dot{x} = Rt\cos t$

$\dot{y} = R\cos t + Rt\sin t - R\cos t$

$\dot{y} = Rt\sin t$



$\int_p 1\, dp = \int_0^2 \sqrt{(Rt\cos t)^2 + (Rt\sin t)^2}\, dt = \int_0^2 \sqrt{R^2t^2(\cos^2 t + \sin^2 t)}\, dt$

$= \int_0^2 Rt\, dt = R\left(\frac{1}{2}t^2\right)\Big|_0^2 = \boxed{2R}$

Problem 5 **18 pts**

- Use vector notations to formulate Coulomb's Law for the electric point-charges Q and q. (Q at origin, q at $P = (x, y, z)$) **2 pts**

$\vec{F}_E = k \frac{Qq}{\|\vec{r}\|^2} \hat{r}$, $r = \langle x, y, z \rangle$, $\|\vec{r}\| = \sqrt{x^2+y^2+z^2}$, $\hat{r} = \frac{1}{\sqrt{x^2+y^2+z^2}} \langle x, y, z \rangle$

$$\boxed{\vec{F}_E = \frac{kQq}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle}$$ (electric force)

$$\boxed{\vec{E} = \frac{\vec{F}_E}{q} = \frac{kQ}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle}$$ (electric field)

- Find the potential $\overset{V}{\cancel{U}}$ an electric charge Q placed at the origin creates at a point $P = (x, y, z)$. Check that the found $\overset{V}{\cancel{U}}$ is the potential by computing $\nabla \overset{V}{\cancel{U}}$. **6 pts**

(electric potential/voltage, NOT electric potential energy)

$V = -k \frac{Q}{\|\vec{r}\|}$

$$\boxed{V = -\frac{kQ}{\sqrt{x^2+y^2+z^2}}}$$ (electric potential/voltage)

$$\nabla V = \left\langle -kQ\left(\frac{-\frac{1}{2}}{(x^2+y^2+z^2)^{3/2}} \cdot 2x\right), -kQ\left(\frac{-\frac{1}{2}}{(x^2+y^2+z^2)^{3/2}} \cdot 2y\right), -kQ\left(\frac{-\frac{1}{2}}{(x^2+y^2+z^2)^{3/2}} \cdot 2z\right) \right\rangle$$

$$\nabla V = \left\langle \frac{kQx}{(x^2+y^2+z^2)^{3/2}}, \frac{kQy}{(x^2+y^2+z^2)^{3/2}}, \frac{kQz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

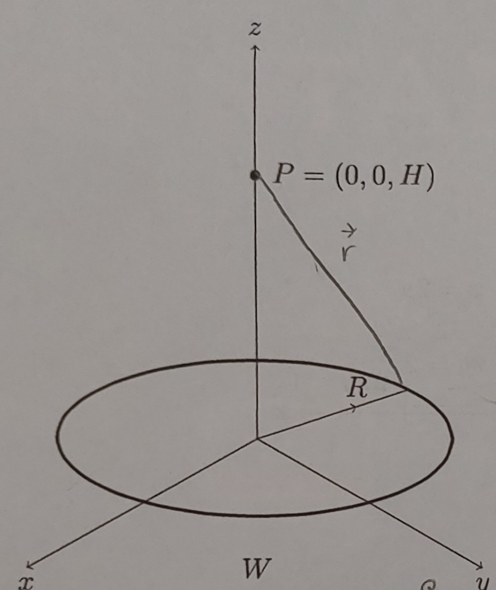
$$\nabla V = \frac{kQ}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$$

$$\boxed{\nabla V = \vec{E}}$$

The problem continues to the next page.

- An electric charge Q is evenly distributed along a thin circular wire of radius R located in the xy-plane and centered at the origin as on the picture below. **10 pts**

Find the potential of the wire at the point $P = (0, 0, H)$.



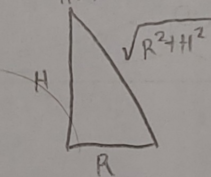
$\mu(x,y,z) = \frac{Q}{2\pi R}$

$\vec{p}(t) = \langle R\cos t, R\sin t, 0\rangle, \quad 0 \le t \le 2\pi$

$\dot{p} = \langle -R\sin t, R\cos t, 0\rangle$

$\|\dot{p}\| = R$

$\|\vec{r}\| = \sqrt{R^2+H^2}$



$V(P) = k\int_P \frac{\mu(x,y,z)}{\|\vec{r}\|}\, dp = k\int_0^{2\pi} \frac{\frac{Q}{2\pi R}}{\sqrt{R^2+H^2}} R\, dt = \frac{kQ}{2\pi\sqrt{R^2+H^2}} \int_0^{2\pi} dt$

$\boxed{V(P) = \frac{kQ}{\sqrt{R^2+H^2}}}$

Problem 6 **10 pts**

Find the area of the region D in the first quadrant bounded by the curves on the picture below. Leave the answer in the symbolic form, but simplify it as much as possible.

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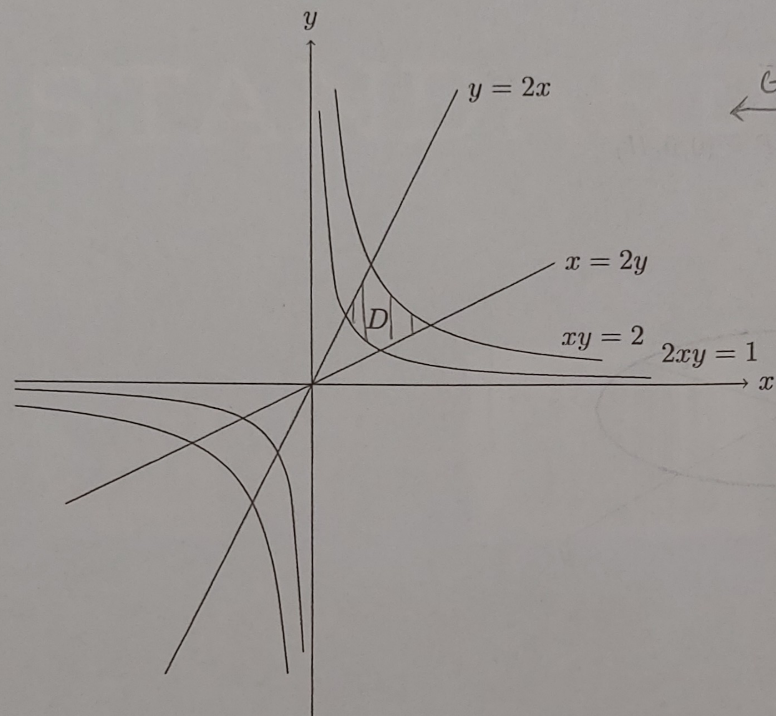
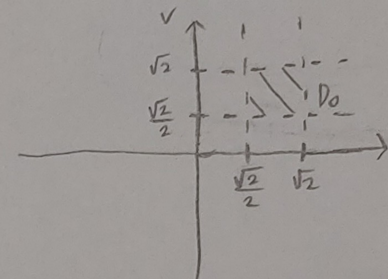


$\xleftarrow{G}$



$\frac{1}{2} \le xy \le 2$ $\quad u = \sqrt{xy}$ $\quad uv = y$

$\frac{1}{2} \le \frac{y}{x} \le 2$ $\quad v = \sqrt{\frac{y}{x}}$ $\quad \frac{u}{v} = x$

$G(u,v) = \left(\frac{u}{v}, uv\right)$

$$J(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$$

$J(G) = \left(\frac{1}{v}\right)u - v\left(-\frac{u}{v^2}\right) = \frac{u}{v} + \frac{u}{v}$

$J(G) = \frac{2u}{v}$

$$\iint_D dxdy = \iint_{D_0} \frac{2u}{v}\, dudv = 2\int_{\sqrt{2}/2}^{\sqrt{2}}\int_{\sqrt{2}/2}^{\sqrt{2}} \frac{u}{v}\, dudv = \boxed{2\int_{\sqrt{2}/2}^{\sqrt{2}} u\, du \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{v}\, dv}$$

$$= 2\int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{v}\left(\frac{1}{2}u^2\right)\Big|_{\sqrt{2}/2}^{\sqrt{2}} dv = \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{v}\left(2 - \frac{1}{2}\right) dv = \frac{3}{2}\int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{v}\, dv$$

$$= \frac{3}{2}\ln|v|\,\Big|_{\sqrt{2}/2}^{\sqrt{2}} = \frac{3}{2}\ln\sqrt{2} - \frac{3}{2}\ln\frac{\sqrt{2}}{2} = \frac{3}{2}\ln\left(\sqrt{2}\cdot\frac{2}{\sqrt{2}}\right) = \boxed{\frac{3}{2}\ln 2}$$

Problem 7 **12 pts**

- Write down the formulas that express the rectangular coordinates x, y, and z as functions of the spherical coordinates ρ, ϕ, and θ. **2 pts**

$$\boxed{\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}}$$

- Find the Jacobian of the coordinate change. **10 pts**

$G(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$

$$J(G) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi \cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$J(G) = \rho^2 \sin\phi \cos^2\phi \cos^2\theta + \rho^2 \sin^3\phi \sin^2\theta + \rho^2 \sin^3\phi \cos^2\theta + \rho^2 \sin\phi \cdot \cos^2\phi \sin^2\theta$

$J(G) = \rho^2 \sin\phi (\cos^2\phi \cos^2\theta + \sin^2\phi \sin^2\theta + \sin^2\phi \cos^2\theta + \cos^2\phi \sin^2\theta)$

$J(G) = \rho^2 \sin\phi (\sin^2\phi (\sin^2\theta + \cos^2\theta) + \cos^2\phi (\sin^2\theta + \cos^2\theta))$

$J(G) = \rho^2 \sin\phi (\sin^2\phi + \cos^2\phi)$

$$\boxed{J(G) = \rho^2 \sin\phi}$$

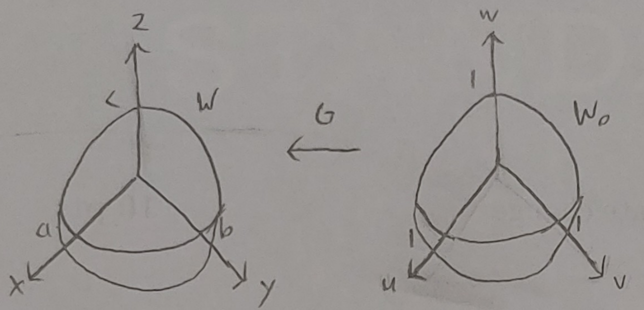
Problem 8 **10 pts**

Find the volume of the following ellipsoid. Hint: a coordinate change helps!

$G(u,v,w) = (au,\ bv,\ cw)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$J(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$



$$\iiint_W dx\,dy\,dz = \iiint_{W_0} abc\, du\,dv\,dw = abc\int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= abc\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \left(\tfrac{1}{3}\rho^3\right)\Big|_0^1 d\phi = \frac{2\pi}{3} abc \int_0^{\pi}\sin\phi\, d\phi$$

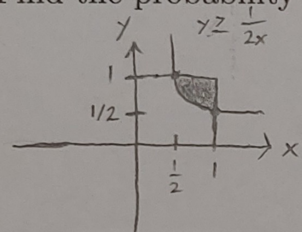
$$= \frac{2\pi}{3} abc\,(-\cos\phi)\Big|_0^{\pi} = \boxed{\frac{4\pi}{3}abc}$$

Problem 9 **10 pts**

The real numbers X and Y are randomly and independently chosen between zero and one. The joint probability density is

$$p(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1]\times[0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability P that the product XY is at least 1/2.



$XY \geq \frac{1}{2}$

$y \geq \frac{1}{2x}$

$P(XY \geq \frac{1}{2}) = \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2x}}^{1} 1 \, dy\,dx = \int_{\frac{1}{2}}^{1} y\Big|_{\frac{1}{2x}}^{1} dx = \int_{\frac{1}{2}}^{1} 1 - \frac{1}{2x} \, dx$

$= \left(x - \frac{1}{2}\ln|x|\right)\Big|_{\frac{1}{2}}^{1} = 1 - \frac{1}{2} - \frac{1}{2}\ln 1 + \frac{1}{2}\ln\frac{1}{2} = \boxed{\frac{1}{2} - \frac{1}{2}\ln 2}$