Course 32 B Sec. 3 | UCLA Department of Mathematics

Winter 2021 | Instructor: Oleg Gleizer

Student: | Student ID:

Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Pr 7	Pr 8	Pr 9	Pr 10	Total
$\overline{}$ 10	$\overline{}$ 10	$\overline{}$ 10	$\overline{}$ 10	$\overline{}$ 10	$\overline{}$ 10	$\overline{}$ 10	$\overline{}$ 10	$\overline{}$ 10	$\overline{}$ 10	$\overline{}$ 100

Midterm 1

Please print your name and student ID in the designated space at the top of the page. Show your work! Answers unsupported by work yield no credit.

Problem 1 Use the symmetry of the problem to find the integral **10 pts**

$$\iint_D \left(1+\frac{y}{x^{10}+y^{10}+10}\right) dA$$

over the domain $D=[-3,2]\times[-1,1]$.

$$\iint_D \left(1+\frac{y}{x^{10}+y^{10}+10}\right) dA =$$

$f(x,y) = -f(x,-y)$

$$\int_{-3}^{2}\int_{-1}^{0} \frac{y}{x^{10}+y^{10}+10}\,dy\,dx = -\int_{-3}^{2}\int_{0}^{1} \frac{y}{x^{10}+y^{10}+10}\,dy\,dx$$

$$= \int_{-3}^{2}\int_{-1}^{1} 1\,dy\,dx \qquad 1\cdot 2\cdot 5 = 10$$

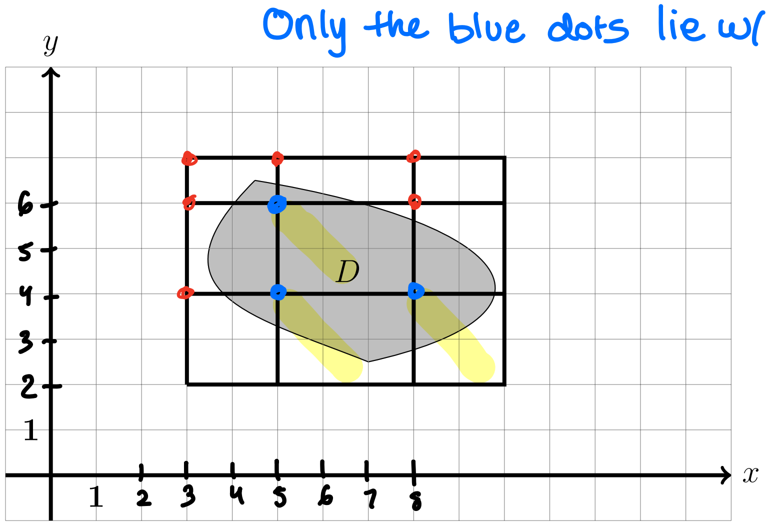
Problem 2 **10 pts**

- Use the upper-left vertices of the below partition to find the Riemann sum $S_{3,3}$ for the integral

$$\iint_D (2x - y)\,dA$$

over the domain D shaded on the picture below. **8 pts**

Only the blue dots lie w/in D



$S_{3,3} =$ $f(5,4) * 6 + f(8,4) * 4 + f(5,6) * 6$

$(2(5)-4) * 6 + (2(8)-4) * 4 + (2(5)-6) * 6$

6 12 4

$36 + 48 + 24$

$\boxed{108}$

$$\begin{array}{r} 36 \\ +\ 48 \\ 24 \\ \hline 108 \end{array}$$

- What is the maximal length $||P||$ of the partition? **2 pts**

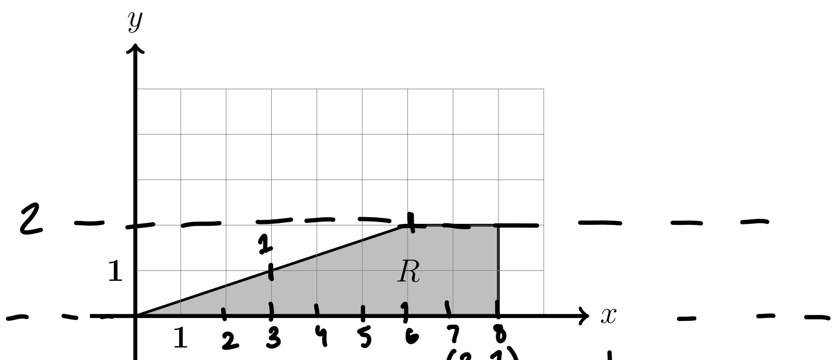
$||P|| =$ 3

Max of Δx & Δy = 3

Problem 3 **10 pts**

Should we consider the below region R as vertically simple or as horizontally simple? Please circle the correct answer. **2 pts**

vertically simple horizontally simple



(3,1) $y = \frac{1}{3}x$ $x = 3y$

(0,0)

Find the following integral. $\iint\limits_R 8xy\,dA =$ **8 pts**

$$\int_0^2 \int_{3y}^8 8xy\,dx\,dy$$

$$8y \int_{3y}^8 x\,dx$$

$$4y \cdot x^2 \Big]_{3y}^8$$

$$4y(64 - 9y^2)$$

$$\int_0^2 (256y - 36y^3)\,dy$$

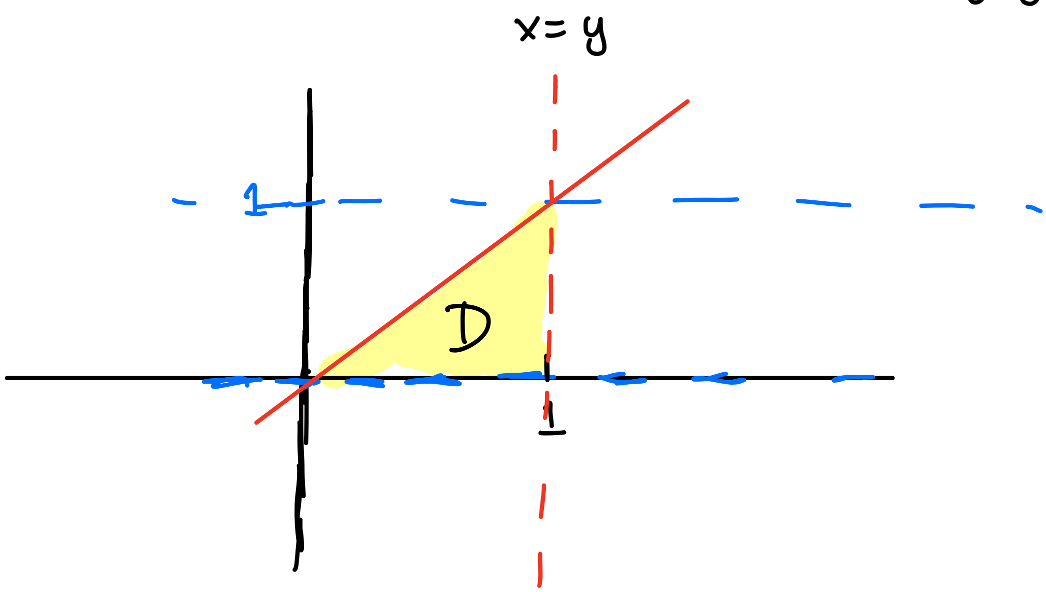
$$128y^2 - 9y^4 \Big]_0^2$$

$$512 - 144 = 368$$

Problem 4 **10 pts**

Evaluate the following integral. Hint: it helps to sketch the domain.

$$\int_0^1 \int_y^1 \frac{\cos x}{x}\, dx dy = \int_0^1 \int_0^x \frac{\cos x}{x}\, dy\, dx$$



$$\frac{\cos x}{x}\, y \Big]_0^x$$

$$\frac{\cos x}{\cancel{x}} \cancel{x} - 0$$

$$\int_0^1 \cos x\, dx$$

$$\sin x \Big]_0^1$$

$$\sin 1$$

Problem 5 **10 pts**

Find the integral

$$\iiint_B 24xy^2z^3 dV$$

over the box $B = [0, a] \times [0, b] \times [0, c]$.

$$24 \int_0^c \int_0^b \int_0^a xy^2z^3 \, dx\,dy\,dz$$

$$24 \int_0^c z^3 dz \int_0^b y^2 dy \int_0^a x \, dx$$

$$\frac{z^4}{4}\Big]_0^c$$

$$\frac{c^4}{4} - 0$$

$$6c^4 \int_0^b y^2 dy \int_0^a x \, dx$$

$$\frac{y^3}{3}\Big]_0^b$$

$$2\cancel{6}c^4 \frac{b^3}{\cancel{3}}$$

$$2b^3c^4 \int_0^a x \, dx$$

$$\frac{x^2}{2}\Big]_0^a$$

$$\cancel{2}b^3c^4 \cdot \frac{a^2}{\cancel{2}}$$

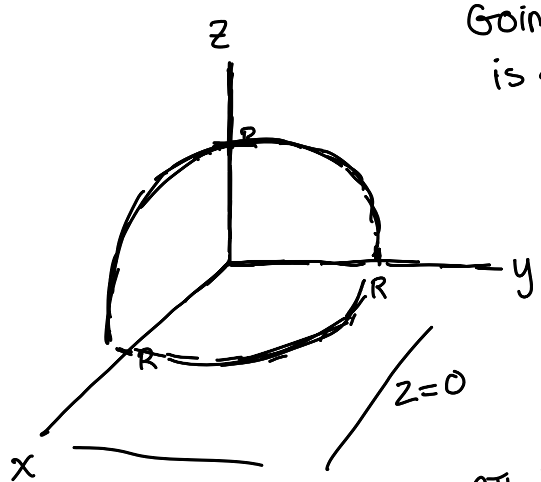
$$a^2b^3c^4$$

Problem 6 **10 pts**

Compute the average value $\bar{f}$ of the function

$$f(x, y, z) = z$$

over the region bounded above by the upper semi-sphere of radius R centered at the origin and bounded below by the plane $z = 0$.



Going to assume "upper semisphere" is synonymous with northern hemisphere.

$$0 \le \theta \le 2\pi$$

$$0 \le \varphi \le \frac{\pi}{2}$$

$$0 \le \rho \le R$$

$$z = \rho\cos\varphi$$

$$\frac{2}{3}\pi R^3$$

$$\frac{\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \rho^3 \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta}{\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}$$

$$\left.\frac{\rho^4}{4}\right]_0^R$$

$$\frac{R^4}{4} \int_0^{2\pi}\int_0^{\pi/2} \cos\varphi \sin\varphi \, d\varphi \, d\theta$$

2π $\pi/2$

Problem 6 cont

$$\frac{\pi R^4}{4}$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\frac{\rho^3}{3}\Big]_0^R$$

$$\frac{R^3}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\varphi \, d\varphi \, d\theta$$

$$-\cos\varphi\Big]_0^{\pi/2}$$

$$0-(-\cos 0)$$

$$+1$$

$$\frac{R^3}{3}\int_0^{2\pi} d\theta$$

$$\frac{2\pi R^3}{3}$$

$$\bar{f} = \frac{3R}{8}$$

$$\frac{R^4}{8}\int_0\int_0 \sin 2\varphi \, d\varphi \, d\theta$$

$$\frac{R^4}{16}\int_0^{2\pi} -\cos 2\varphi\Big]_0^{\pi/2} d\theta$$

$$-\cos\pi - (-\cos 0)$$

$$+1+1$$

$$1+1=2$$

$$\frac{R^4}{8}\int_0^{2\pi} d\theta$$

$$\frac{\cancel{2}^{1}\pi R^4}{\cancel{8}_4}$$

$$\frac{\frac{\pi R^4}{4}\cdot 3}{\frac{2\pi R^3}{3}\cdot 3}$$

$$\frac{3\cancel{\pi}\cancel{R^4}^{R}}{8\cancel{\pi}\cancel{R^3}}$$

Problem 7 **10 pts**

Convert the following equation
to an equation in rectangular coordinates. **5 pts**

$$r = 2a\sin\theta$$

$r \cdot r = 2a\sin\theta \cdot r$ $\qquad$ $r^2 = 2a \cdot r\sin\theta$

$x^2 + y^2 = 2a \cdot y$

$x^2 + y^2 - 2ay + a^2 = a^2$

$x^2 + (y-a)^2 = a^2$

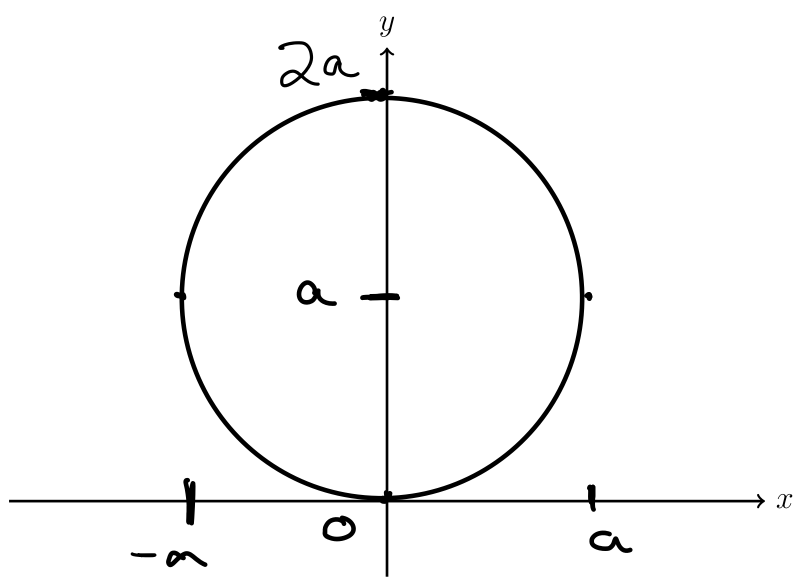
$r^2 = 2a\, r\sin\theta$

$r^2 = 2a \cdot y$

$x^2 + y^2 - 2ay = 0$

$x^2 + y^2 - 2ay + a^2 = a^2$

Graph the set of the points $(\theta, r(\theta))$ that satisfy the above equation. **5 pts**



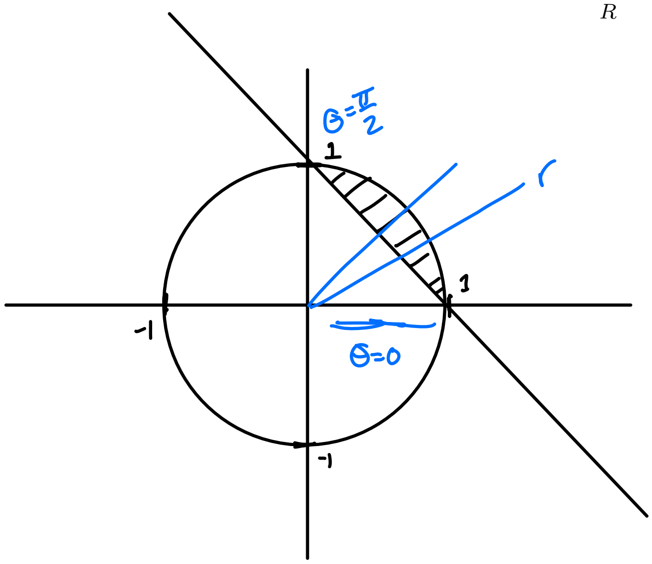
Problem 8 **10 pts**

The region of integration R is given by the following inequalities.

$$R: \quad x^2 + y^2 \leq 1, \quad x + y \geq 1$$

Evaluate the integral below by switching to polar coordinates.

$$\iint_R (x - y)\, dA$$



$$x^2 + y^2 = 1 \qquad\qquad x + y = 1$$

$$r^2 = 1 \qquad\qquad r(\cos\theta + \sin\theta) = 1$$

$$r = 1 \qquad\qquad r = \frac{1}{\cos\theta + \sin\theta}$$

$$f(x,y) = (x - y)$$

$$f(r,\theta) = r\cos\theta - r\sin\theta$$

$$r(\cos\theta - \sin\theta)$$

$$0 \leq \theta \leq \frac{\pi}{2}$$

$$\frac{1}{\cos\theta + \sin\theta} \leq r \leq 1$$

$$\int_0^{\frac{\pi}{2}} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^2(\cos\theta - \sin\theta)\, dr\, d\theta$$

$$\int_0^{\pi/2} \frac{(\cos\theta - \sin\theta)}{3} \Big[r^3 \Big]_{\frac{1}{\cos\theta + \sin\theta}}^{1} d\theta$$

$$\frac{\cos\theta - \sin\theta}{3} \left(1 - \frac{1}{(\cos\theta + \sin\theta)^3} \right)$$

$$\frac{1}{3}\left[\cos\theta - \sin\theta - \frac{\cos\theta-\sin\theta}{(\cos\theta+\sin\theta)^3}\cdot\frac{\cos\theta+\sin\theta}{\cos\theta+\sin\theta}\right.$$

$$\frac{1}{3}\left[\cos\theta - \sin\theta - \frac{\overbrace{\cos^2\theta-\sin^2\theta}^{\cos 2\theta}}{(\cos\theta+\sin\theta)^4}\right]$$

$$\left((\cos\theta+\sin\theta)^2\right)^2$$

$$\left(\cos^2\theta+\sin 2\theta+\sin^2\theta\right)^2$$

$$(1+\sin 2\theta)^2$$

$$\frac{1}{3}\int_0^{\pi/2}(\cos\theta-\sin\theta)\,d\theta - \frac{1}{3}\int_0^{\pi/2}\frac{\cos 2\theta}{(1+\sin 2\theta)^2}\,d\theta$$

$$\frac{1}{3}\left(\sin\theta+\cos\theta\Big]_0^{\pi/2}\right) - \frac{1}{3}\int_0^{\pi/2}\frac{\cos 2\theta}{(1+\sin 2\theta)^2}\,d\theta$$

$$\frac{1}{3}\left(\sin\tfrac{\pi}{2}+\cos\tfrac{\pi}{2}-\sin\varnothing-\cos\varnothing\right)$$

$$1\cancel{+0-0}-1$$

$$\varnothing$$

$$u = 1+\sin 2\theta$$

$$\frac{du}{d\theta} = 2\cos 2\theta$$

$$d\theta = \frac{du}{2\cos 2\theta}$$

$$u = 1+\sin\varnothing = 1$$

$$u = 1+\sin\pi = 1$$

$$\varnothing - \frac{1}{6}\int_1^1 \frac{1}{u^2}\,du$$

$$\varnothing$$

$$\varnothing$$

Problem 9 **10 pts**

The region of integration R is given by the following inequalities.

$$R: \quad x^2 + y^2 \le 1, \quad x \ge 0, \quad y \ge 0, \quad 0 \le z \le 2$$

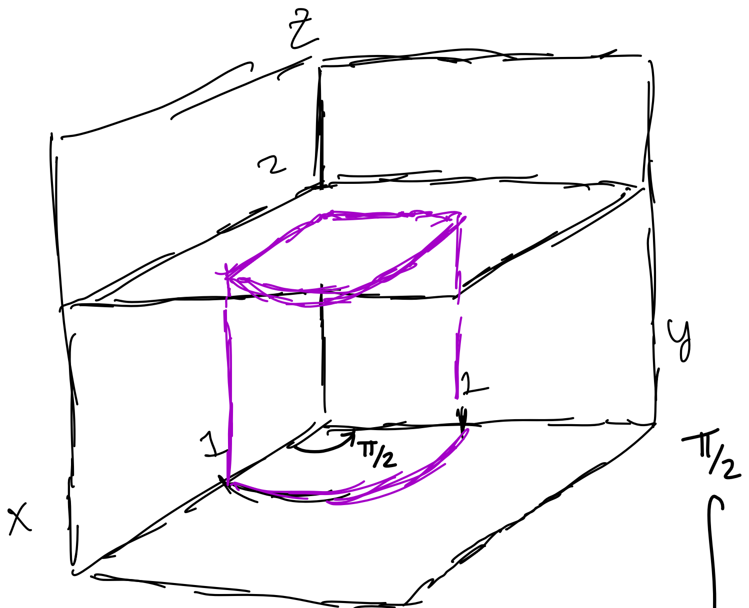
Use cylindrical coordinates to evaluate the below integral.

$$\iiint_R x\,dV$$

$2\pi r^2 h$

$\frac{4\pi}{4}$

1/4 of a cylinder



$0 \le \theta \le \frac{\pi}{2}$

$0 \le r \le 1$

$0 \le z \le 2$

$x = r\cos\theta$

$$\int_0^{\pi/2}\int_0^1\int_0^2 r^2\cos\theta \, dz\, dr\, d\theta$$

$$r^2\cos\theta\, z \Big]_0^2$$

$2r^2\cos\theta$

$$\int_0^{\pi/2}\int_0^1 2r^2\cos\theta\, dr\, d\theta$$

$$\frac{2r^3\cos\theta}{3}\Big]_0^1$$

$2r^2\cos\theta$

$\frac{2r^3\cos\theta}{3}$

$\frac{2\cos\theta}{3}$

$$\int^{\pi/2} \frac{2\cos\theta}{} \, d\theta$$

$$\int_0 \frac{\quad}{3}$$

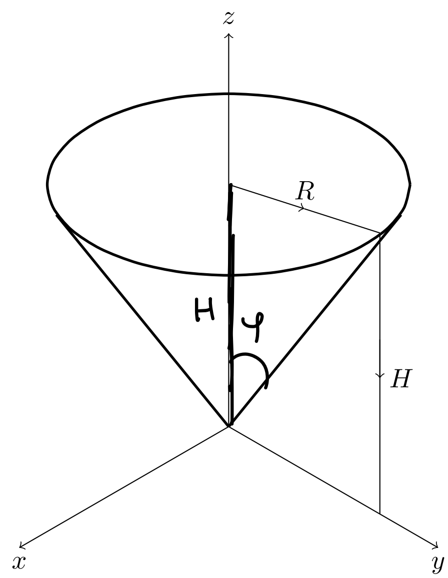
$$\frac{2}{3}\sin\theta \Big]_0^{\pi/2}$$

$$\frac{2}{3}$$

Problem 10 **10 pts**

Use spherical coordinates to figure out the volume of the below cone.

$Z = H$



$\frac{\pi r^2 H}{3}$

$0 \leq \theta \leq 2\pi$

$\varphi = \arctan \frac{R}{H}$

$0 \leq \rho \leq \frac{H}{\cos\varphi}$

$z = \rho\cos\varphi$

$z = \rho\cos\varphi$

$H = \rho\cos\varphi$

$\rho = \frac{H}{\cos\varphi}$

$$\int_0^{2\pi}\int_0^{\arctan(\frac{R}{H})}\int_0^{\frac{H}{\cos\varphi}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\frac{\rho^3 \sin\varphi}{3}\Big]_0^{\frac{H}{\cos\varphi}}$$

$$\int_0^{2\pi}\int_0^{\arctan(\frac{R}{H})} \frac{H^3 \sin\varphi}{3\cos^3\varphi}$$

$$\frac{H^3}{3} \cdot \tan\varphi \sec^2\varphi \, d\varphi \, d\theta$$

$$H^3 \int^{2\pi}\int^{\arctan(\frac{R}{H})} \tan\varphi \sec^2\varphi \, d\varphi \, d\theta$$

$$\frac{\quad}{3}\int_0 \int_0 \tan\varphi \sec^2\varphi \, d\varphi \, d\theta$$

$u = \tan\varphi$

$\frac{du}{d\varphi} = \sec^2\varphi$

$d\varphi = \frac{du}{\sec^2\varphi}$ $\qquad \tan(\arctan(\frac{R}{H}))$

$$\frac{H^3}{3}\int_0^{2\pi}\int_0^{\frac{R}{H}} u \, du \quad d\theta$$

$$\frac{H^3}{6}\int_0^{2\pi} u^2 \Big]_0^{\frac{R}{H}} \, d\theta$$

$$\frac{H^3}{6}\int_0^{2\pi} \frac{R^2}{H^2} \, d\theta$$

$$\frac{R^2}{\cancel{H^2}} \cdot \frac{\cancel{H^3}H}{6}\int_0^{2\pi} d\theta$$

$$\frac{HR^2}{6}\left(\theta\right]_0^{2\pi}$$

$$\frac{\pi R^2 H}{3} \qquad \frac{{}^1\cancel{2}\pi HR^2}{\cancel{6}\,3}$$