Course 32 B Sec. 2 | UCLA Department of Mathematics

Fall 2020 | Instructor: Oleg Gleizer

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Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Pr 7	Pr 8	Pr 9	Pr 10	Total
$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{100}$

Midterm 1

Please print your name and student ID in the designated space at the top of the page. Show your work! Answers unsupported by work yield no credit.

Problem 1 Use the symmetry of the problem to find the integral **10 pts**

$$\iint_D \left(1 + \frac{y}{x^{10} + y^{10} + 10}\right) dA$$

over the domain $D = [-3, 2] \times [-1, 1]$.

$$\iint_D \left(1 + \frac{y}{x^{10} + y^{10} + 10}\right) dA = \iint 1 \, dA + \iint \frac{y}{x^{10}+y^{10}+10} \, dA$$

$f(x,y) = \dfrac{y}{x^{10}+y^{10}+10}$ $\quad f(x,-y) = \dfrac{-y}{x^{10}+y^{10}+10} = -f(x,y)$

$\int_{-1}^{1}\int_{-3}^{2} \frac{y}{x^{10}+y^{10}+10} \, dx\,dy = 0$ by symmetry

$\int_{-1}^{1}\int_{-3}^{2} 1 \, dx\,dy = \int_{-1}^{1} 5 \, dy$

$= 5 - (-5)$

$= 10$

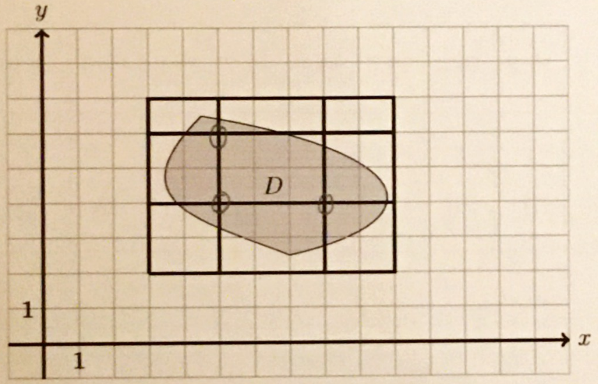
10

Problem 2 **10 pts**

- Use the upper-left vertices of the below partition to find the Riemann sum $S_{3,3}$ for the integral

$$\iint_D (2x - y)dA$$

over the domain D shaded on the picture below. **8 pts**



$S_{3,3} = f(5,4)\cdot 3\cdot 2 + f(5,6)\cdot 3\cdot 2 + f(8,4)\cdot 2\cdot 2$

$= 36 + 24 + 48$

$= \boxed{108}$

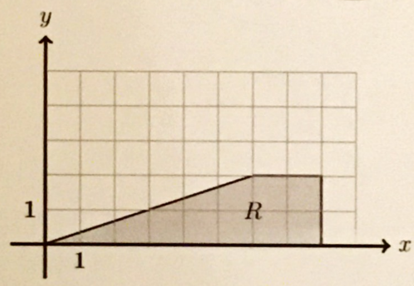
- What is the maximal length $||P||$ of the partition? **2 pts**

$||P|| = \boxed{3}$

Problem 3 **10 pts**

Should we consider the below region R as vertically simple or as horizontally simple? Please circle the correct answer. **2 pts**

vertically simple horizontally simple



Find the following integral. $\iint_R 8xy\,dA =$ **8 pts**

$0 \le y \le 2 \qquad y = \frac{1}{3}x \Rightarrow x = 3y \qquad x = 8$

$$\int_0^2 \int_{3y}^8 8xy\,dx\,dy$$

$$\int_{3y}^8 8xy\,dx = 8y\int_{3y}^8 x\,dx = 8y\left[\frac{x^2}{2}\right]_{3y}^8 = 8y\left(32 - \frac{9y^2}{2}\right) = 256y - 36y^3$$

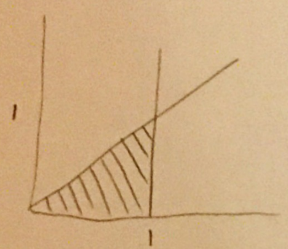
$$\int_0^2 256y - 36y^3\,dy = \left[128y^2 - 9y^4\right]_0^2 = 512 - 144 = \boxed{368}$$

Problem 4 **10 pts**

Evaluate the following integral. Hint: it helps to sketch the domain.

$$\int_0^1 \int_y^1 \frac{\cos x}{x}\, dx dy$$

$0 \leq y \leq 1 \qquad y \leq x \leq 1$



$0 \leq x \leq 1$

$0 \leq y \leq x$

$$\int_0^1 \int_0^x \frac{\cos x}{x}\, dy\, dx$$

$$\int_0^x \frac{\cos x}{x}\, dy = \frac{\cos x}{x} \int_0^x 1\, dy = \frac{\cos x}{x} \left[y \right]_0^x = \frac{\cos x}{x}(x) = \cos x$$

$$\int_0^1 \cos x\, dx = \left[\sin x \right]_0^1 = \sin(1) - \sin(0) = \boxed{\sin(1)}$$

Problem 5 **10 pts**

Find the integral

$$\iiint_B 24xy^2z^3 dV$$

over the box $B = [0, a] \times [0, b] \times [0, c]$.

$$\int_0^c \int_0^b \int_0^a 24xy^2z^3 \, dx\,dy\,dz$$

$$\int_0^a x\,dx = \left[\frac{x^2}{2}\right]_0^a = \frac{a^2}{2}$$

$$\int_0^b y^2\,dy = \left[\frac{y^3}{3}\right]_0^b = \frac{b^3}{3}$$

$$\int_0^c z^3\,dz = \left[\frac{z^4}{4}\right]_0^c = \frac{c^4}{4}$$

$$24\left(\frac{a^2}{2}\right)\left(\frac{b^3}{3}\right)\left(\frac{c^4}{4}\right) = \boxed{a^2b^3c^4}$$

Problem 6 **10 pts**

Compute the average value $\bar{f}$ of the function

$$f(x, y, z) = z$$

over the region bounded above by the upper semi-sphere of radius R centered at the origin and bounded below by the plane $z = 0$.

$f(x,y,z) = z = \rho\cos\phi \qquad 0 \leq \theta \leq 2\pi \quad 0 \leq \rho \leq R \quad 0 \leq \phi \leq \pi/2$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \rho\cos\phi\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \iiint (\rho^3 \sin\phi \cos\phi)\, d\rho\, d\phi\, d\theta$$

$$\int_0^{2\pi} 1\, d\theta = [\theta]_0^{2\pi} = 2\pi$$

$$\int_0^{\pi/2} \sin\phi\cos\phi\, d\phi = \left[\tfrac{1}{2}(\sin\phi)^2\right]_0^{\pi/2} = 1/2$$

$\frac{d}{d\theta}(\sin\theta)^2 = 2\sin\theta\cos\theta$

$$\int_0^{R} \rho^3\, d\rho = \left[\frac{\rho^4}{4}\right]_0^{R} = \frac{R^4}{4}$$

$$(2\pi)(1/2)\left(\frac{R^4}{4}\right) = \frac{\pi R^4}{4}$$

$$\text{Volume} = \frac{(4/3\,\pi R^3)}{2} = 2/3\,\pi R^3$$

$$\text{Average value} = \left(\frac{\pi R^4}{4}\right) \div \left(\frac{2\pi R^3}{3}\right) = \frac{3\pi R^4}{8\pi R^3} = \boxed{\frac{3R}{8}}$$

Problem 7 **10 pts**

Convert the following equation to an equation in rectangular coordinates. **5 pts**

$$r = 2a\sin\theta$$

$r^2 = 2ar\sin\theta$

$x^2+y^2 = 2ay$

$x^2+y^2-2ay=0$

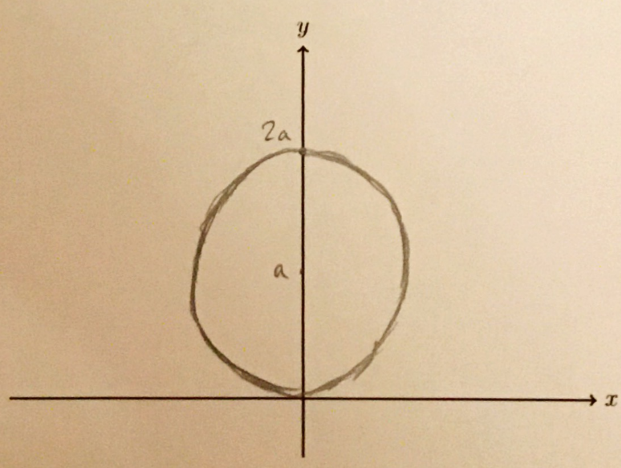
$x^2+(y-a)^2-a^2=0$

$\boxed{x^2+(y-a)^2=a^2}$

$x^2+y^2=r^2$

$y=r\sin\theta$

Graph the set of the points $(\theta, r(\theta))$ that satisfy the above equation. **5 pts**



Center $(0,a)$

Radius a

$x+y \geq 1$
$y \geq 1-x$

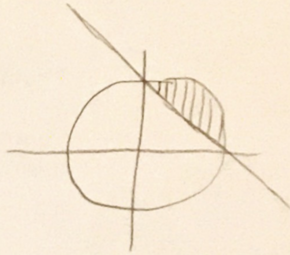
Problem 8 **10 pts**

The region of integration R is given by the following inequalities.

$$R: \; x^2+y^2 \leq 1, \; x+y \geq 1$$

Evaluate the integral below by switching to polar coordinates.

$$\iint_R (x-y)dA$$



$y = 1-x$

$x^2+y^2=1$

$x^2+(1-x)^2=1$

$x^2+1-2x+x^2=1$

$2x^2-2x=0$

$2x(x-1)=0 \quad x=0 \text{ or } 1$

$x=0 \Rightarrow y=1 \quad x=1 \Rightarrow y=0$

Points of intersection: $(0,1)$, $(1,0)$

$0 \leq \theta \leq \pi/2$

$x^2+y^2=1 \Rightarrow r=1$

$x+y=1 \Rightarrow r\cos\theta + r\sin\theta = 1$

$r(\cos\theta+\sin\theta)=1$

$r = \frac{1}{\cos\theta+\sin\theta}$

$f(x,y) = x-y = r\cos\theta - r\sin\theta$
$= r(\cos\theta - \sin\theta)$

$$\int_0^{\pi/2}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r(\cos\theta-\sin\theta)\,r\,dr\,d\theta = \iint r^2(\cos\theta-\sin\theta)\,dr\,d\theta$$

$$\int_{\frac{1}{c+s}}^{1} r^2(c-s)\,dr = (c-s)\int_{\frac{1}{c+s}}^{1} r^2\,dr = (c-s)\left[\frac{r^3}{3}\right]_{\frac{1}{c+s}}^{1} = (c-s)\left(\frac{1}{3}-\frac{1}{3(c+s)^3}\right)$$
$$= \frac{c-s}{3} - \frac{c-s}{3(c+s)^3}$$

$$\int_0^{\pi/2}\frac{c-s}{3}\,d\theta - \int_0^{\pi/2}\frac{c-s}{3(c+s)^3}\,d\theta$$

$$\int_0^{\pi/2}\frac{c-s}{3}\,d\theta = \frac{1}{3}\int_0^{\pi/2}\cos\theta-\sin\theta\,d\theta = \frac{1}{3}\left[\sin\theta+\cos\theta\right]_0^{\pi/2} = \frac{1}{3}(1-1)$$
$$= 0$$

$$\int_0^{\pi/2}\frac{c-s}{3(c+s)^3}\,d\theta = \frac{1}{3}\int_0^{\pi/2}\frac{\cos\theta-\sin\theta}{(\cos\theta+\sin\theta)^3}\,d\theta = \frac{1}{3}\int_1^1\frac{1}{u^3}\,du$$
$$= 0$$

$u = \sin\theta+\cos\theta$
$du = \cos\theta - \sin\theta\, d\theta$
$\theta = \pi/2 \rightarrow u = 1$
$\theta = 0 \rightarrow u = 1$

$$0 - 0 = \boxed{0}$$

Problem 9 **10 pts**

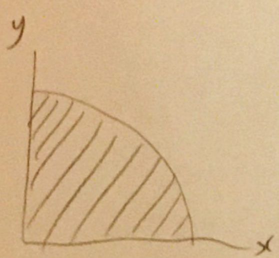
The region of integration R is given by the following inequalities.

$$R: \; x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 2$$

Use cylindrical coordinates to evaluate the below integral.

$$\iiint_R x\,dV$$

$x^2 + y^2 = 1$
radius $= 1$



$0 \leq \theta \leq \pi/2$
$0 \leq r \leq 1$
$0 \leq z \leq 2$

$f(x,y,z) = x = r\cos\theta$

$\int_0^{\pi/2} \int_0^1 \int_0^2 r\cos\theta \, r \, dz\,dr\,d\theta = \iiint (r^2 \cos\theta)\,dz\,dr\,d\theta$

$\int_0^{\pi/2} \cos\theta \, d\theta = [\sin\theta]_0^{\pi/2} = 1$

$\int_0^1 r^2\,dr = [\frac{r^3}{3}]_0^1 = 1/3$

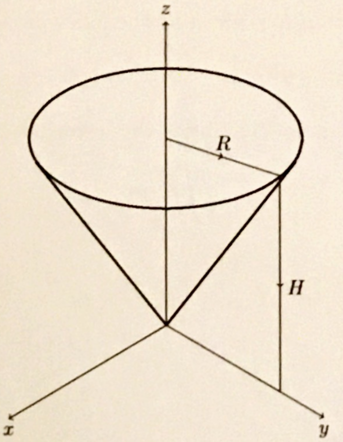
$\int_0^2 1\,dz = [z]_0^2 = 2$

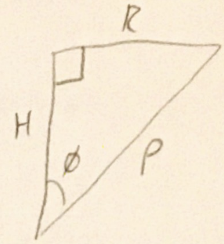
$(1)(1/3)(2) = \boxed{2/3}$

Problem 10 **10 pts**

Use spherical coordinates to figure out the volume of the below cone.

$$\text{Volume} = \iiint_W 1 \, dV$$





$$\tan\phi = R/H$$

$$\phi = \tan^{-1}(R/H)$$

$$\cos\phi = H/\rho$$

$$\rho = H/\cos\phi$$

$$0 \le \theta \le 2\pi$$

$$0 \le \phi \le \tan^{-1}(R/H)$$

$$0 \le \rho \le H/\cos\phi$$

$$\int_0^{2\pi} \int_0^{\tan^{-1}(R/H)} \int_0^{H/\cos\phi} (1)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{H/\cos\phi} \rho^2 \sin\phi \, d\rho = \sin\phi \int_0^{H/\cos\phi} \rho^2 d\rho = \sin\phi \left[\frac{\rho^3}{3} \right]_0^{H/\cos\phi} = \tfrac{1}{3} \sin\phi \, \frac{H^3}{\cos^3\phi}$$

$$= \tfrac{1}{3} H^3 \sin\phi \, \frac{1}{\cos\phi} \sec^2\phi$$

$$= \tfrac{1}{3} H^3 \tan\phi \sec^2\phi$$

$$\int_0^{\tan^{-1}(R/H)} \tfrac{1}{3} H^3 \tan\phi \sec^2\phi \, d\phi$$

$$= \tfrac{1}{3} H^3 \int_0^{\tan^{-1}(R/H)} \tan\phi \sec^2\phi \, d\phi = \tfrac{1}{3} H^3 \left[\tfrac{1}{2} (\tan\phi)^2 \right]_0^{\tan^{-1}(R/H)}$$

$$\frac{d}{d\theta} (\tan\theta)^2 = 2 \tan\theta \sec^2\theta$$

$$= \tfrac{1}{6} H^3 \left(\frac{R^2}{H^2} \right) = \tfrac{1}{6} H R^2$$

$$\int_0^{2\pi} \tfrac{1}{6} H R^2 \, d\theta = \tfrac{1}{6} H R^2 \left[\theta \right]_0^{2\pi} = \boxed{\tfrac{1}{3} \pi R^2 H}$$