Course 32 B Sec. 2 | UCLA Department of Mathematics

Fall 2020 | Instructor: Oleg Gleizer

Student: | Student ID:

Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Pr 7	Pr 8	Pr 9	Pr 10	Total
$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{100}$

Midterm 1

Please print your name and student ID in the designated space at the top of the page. Show your work! Answers unsupported by work yield no credit.

Problem 1 Use the symmetry of the problem to find the integral **10 pts**

$$\iint\limits_D \left(1 + \frac{y}{x^{10} + y^{10} + 10}\right) dA$$

over the domain $D = [-3, 2] \times [-1, 1]$.

$$\iint\limits_D \left(1 + \frac{y}{x^{10} + y^{10} + 10}\right) dA = \iint_D 1\,dA + \iint_D \frac{y}{x^{10}+y^{10}+10}\,dA. = I$$

Note that if $f(x,y) = \frac{y}{x^{10}+y^{10}+10}$, then $f(x,-y) = -f(x,y)$. Moreover, our domain is symmetric along the y axis. So for every point in domain, there exist a point in the polar opposite y coordinate which has a value equal in magnitude but opposite in sign for our function. So $\iint_D f(x,y)\,dA = 0$.

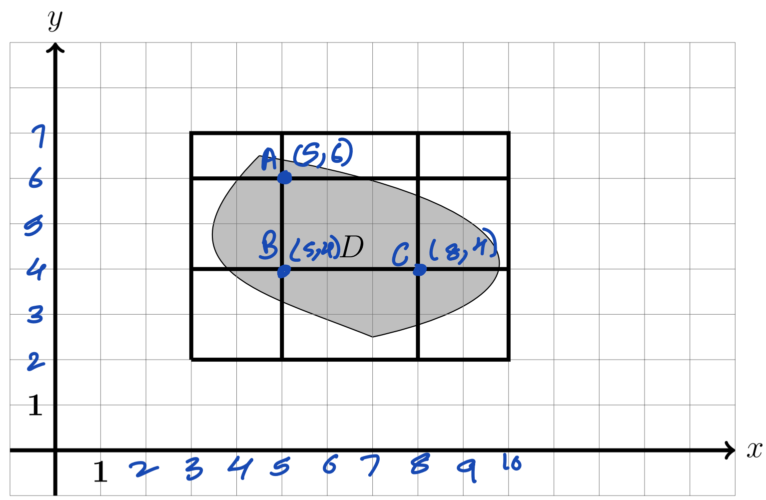
I = Area of domain $+ 0 = |(-3-2)\times(-1-1)| = 10.$

Problem 2 **10 pts**

- Use the upper-left vertices of the below partition to find the Riemann sum $S_{3,3}$ for the integral

$$\iint_D (2x - y)\,dA$$

over the domain D shaded on the picture below. **8 pts**



we consider a point only if it lies in domain. All other points have a value of 0.

$f(x,y) = 2x - y.$

$S_{3,3} = f(A)\cdot \text{areaOfRectAt}(A) + f(B)\cdot \text{areaOfRectAt}(B) + f(C)\cdot \text{areaOfRectAt}(C)$

$= f(5,6)\cdot(8-5)\cdot(6-4) + f(5,4)\cdot(8-5)(4-2) + f(8,4)\cdot(10-8)(4-2)$

$= 4\cdot 3\cdot 2 + 6\cdot 3\cdot 2 + 12\cdot 2\cdot 2 = 24 + 36 + 48$

$= 108$

- What is the maximal length $||P||$ of the partition? **2 pts**

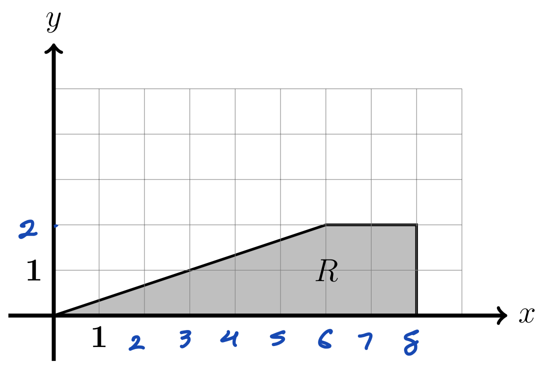
$||P|| = 3$ [max dimension of a partition]

since $||P|| = \max\limits_{\substack{1\le i\le N \\ 1\le j\le m}} (x_i - x_{i-1},\ y_j - y_{j-1})$

Problem 3 **10 pts**

Should we consider the below region R as vertically simple or as horizontally simple? Please circle the correct answer. **2 pts**

vertically simple horizontally simple



Find the following integral. $\iint_R 8xy\,dA =$ **8 pts**

we can take the lower bound as $f(y)=3y$ and the upper bound as $g(y)=8$ for x.

The $0 \le y \le 2$.

Doing this integral:

$$\iint_R 8xy\,dA = \int_{y=0}^{2}\int_{x=3y}^{8} 8xy\,dx\,dy = \int_{y=0}^{2} 4x^2y\Big|_{x=3y}^{8} = \int_{y=0}^{2}(256y-36y^3)\,dy$$

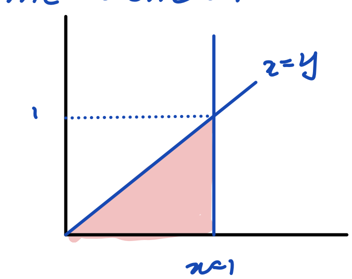
$$= \left(128y^2-9y^4\right)_0^2 = 512-168 = 368$$

Problem 4 **10 pts**

Evaluate the following integral. Hint: it helps to sketch the domain.

$$\int_0^1 \int_y^1 \frac{\cos x}{x}\, dx dy$$

The domain can be drawn as



We can try integrating over y first, so changing the order of integration we have that

$$\int_{x=0}^{1} \int_{y=0}^{x} \frac{\cos x}{x}\, dy dx = \int_{x=0}^{1} \frac{\cos x}{x}\, y \Big|_{y=0}^{x} dx$$

$$= \int_{x=0}^{1} \frac{\cos x}{x} \cdot x \, dx = \int_0^1 \cos x \, dx$$

$$= [\sin x]_0^1 = \sin 1$$

The reason we did change of order of integration was to make the integration easier. As can be seen, instead of having to integrate over $\frac{\cos x}{x} dx$ we just had to integrate over $\cos x\, dx$ since integrating with y first allowed us to remove x from the denominator.

Problem 5 **10 pts**

Find the integral

$$\iiint_B 24xy^2z^3 dV$$

over the box $B = [0,a] \times [0,b] \times [0,c]$.

$$\iiint_B 24xy^2z^3 dV = \int_0^a\int_0^b\int_0^c 24xy^2z^3 dz dy dx = \int_0^a\int_0^b 6xy^2z^4\Big|_{z=0}^{c} dy dx$$

$$= \int_0^a\int_0^b 6c^4xy^2 dy dx = \int_0^a\int_0^b 2c^4xy^3\Big|_{y=0}^{b} dx = \int_0^a 2c^4b^3x\, dx$$

$$= c^4b^3\left[x^2\right]_0^a = a^2b^3c^4$$

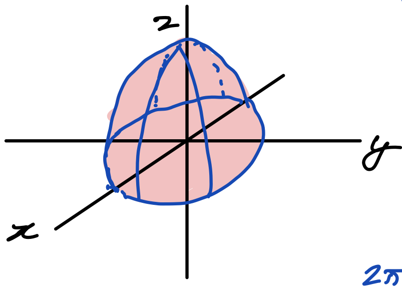
Problem 6 **10 pts**

Compute the average value $\bar{f}$ of the function

$$f(x, y, z) = z$$

over the region bounded above by the upper semi-sphere of radius R centered at the origin and bounded below by the plane $z = 0$.

We can draw the region as follows:



Converting to spherical coordinates, we have

$\rho : 0 \to R$

$\phi : 0 \to \frac{\pi}{2}$ [It is a hemisphere]

$\theta : 0 \to 2\pi$

So, $$\iiint_R z \cdot dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}}\int_{\rho=0}^{R} \rho\cos\phi \cdot (\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta) = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}} \frac{\rho^4}{4}\sin\phi\cos\phi \Big|_{\rho=0}^{R} d\phi \, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}} \frac{R^4}{4}\sin\phi\cos\phi \, d\phi \, d\theta = \frac{R^4}{8}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}} 2\sin\phi\cos\phi \, d\phi \, d\theta$$

$$= \frac{R^4}{8}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}} \sin 2\phi \, d\phi \, d\theta = \frac{R^4}{8}\int_{\theta=0}^{2\pi}\left[\frac{-\cos 2\phi}{2}\right]_0^{\frac{\pi}{2}} d\theta = \frac{R^4}{8}\int_{\theta=0}^{2\pi} 1 \cdot d\theta = \frac{\pi R^4}{4}$$

We can calculate $\iiint_R dV$ by just calculating the volume of a hemisphere with radius R, which we know to be $\frac{2}{3}\pi R^3$

The average value is defined as $$\frac{\iiint f(x,y,z)\,dV}{\iiint dV} = \frac{\frac{\pi R^4}{4}}{\frac{2}{3}\pi R^3} = \frac{3}{8}R$$

Problem 7 **10 pts**

Convert the following equation
to an equation in rectangular coordinates. **5 pts**

$$r = 2a\sin\theta$$

Multiplying both sides by r

$$r^2 = 2ar\sin\theta$$

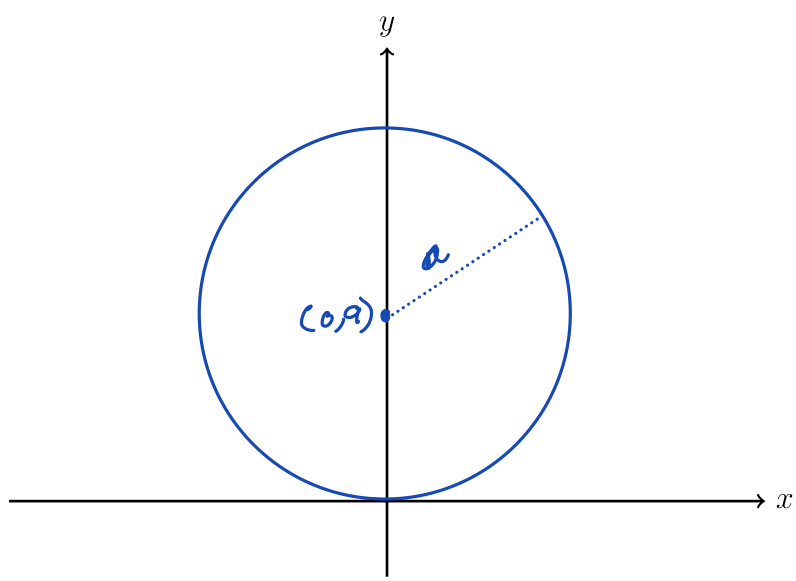
We know that $r^2 = x^2 + y^2$, $r\sin\theta = y$. Making these replacements we get

$$x^2 + y^2 = 2ay$$

$$x^2 + y^2 - 2ay + a^2 = a^2$$

$$x^2 + (y-a)^2 = a^2$$

Graph the set of the points $(\theta, r(\theta))$ that satisfy the above equation. **5 pts**



Problem 8 **10 pts**

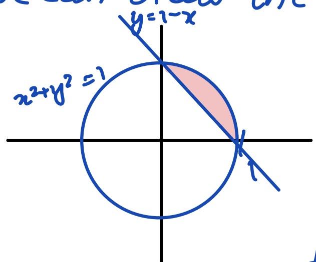
The region of integration R is given by the following inequalities.

$$R: \quad x^2 + y^2 \leq 1, \quad x + y \geq 1$$

Evaluate the integral below by switching to polar coordinates.

$$\iint_R (x - y) dA$$

First, we can draw the domain of integration.



Converting to polar coordinates, the upper bound for r is $r \leq 1$ [as $x^2+y^2 \leq 1$].

The lower bound must be converted from $x+y \geq 1 \Rightarrow r\sin\theta + r\cos\theta \geq 1$

[In the domain, $\theta \in [0, \frac{\pi}{2}]$ so $\sin\theta, \cos\theta \geq 0$.] $\Rightarrow r \geq \dfrac{1}{\sin\theta+\cos\theta}$.

So we get that $r: \frac{1}{\sin\theta+\cos\theta} \to 1$; $\theta: 0 \to \frac{\pi}{2}$.

Moreover, $f(x,y) = x - y \Leftrightarrow f(r,\theta) = r(\cos\theta - \sin\theta)$

$$\iint_R (x-y)\,dA = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=\frac{1}{\sin\theta+\cos\theta}}^{1} r(\cos\theta - \sin\theta)\, r\,dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \frac{r^3}{3}(\cos\theta - \sin\theta)\Big|_{r=\frac{1}{\sin\theta+\cos\theta}}^{1} d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{\cos\theta - \sin\theta}{3}\,d\theta - \int_{\theta=0}^{\frac{\pi}{2}} \frac{\cos\theta - \sin\theta}{3(\sin\theta+\cos\theta)^3}\,d\theta$$

$$= \frac{1}{3}\left[\sin\theta + \cos\theta\right]_0^{\frac{\pi}{2}} - I$$

$$= \frac{1}{3}[1 + 0 - 0 - 1]$$

$$= 0$$

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \frac{\cos\theta - \sin\theta}{3(\sin\theta+\cos\theta)^3}\,d\theta$$

$t = \sin\theta + \cos\theta$

$dt = \cos\theta - \sin\theta \, d\theta$

$\theta = 0 \Rightarrow t = 1$

$\theta = \frac{\pi}{2} \Rightarrow t = 1$

$$I = \int_1^1 \frac{dt}{3t^3} = 0$$

Problem 9 **10 pts**

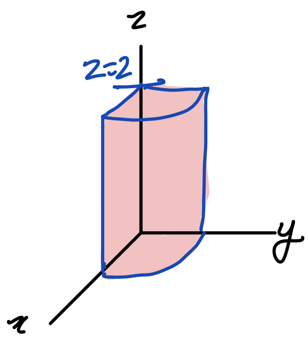
The region of integration R is given by the following inequalities.

$$R: \quad x^2 + y^2 \le 1, \quad x \ge 0, \quad y \ge 0, \quad 0 \le z \le 2$$

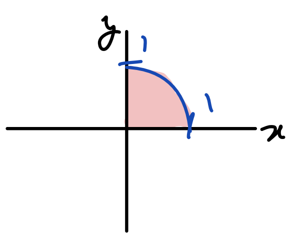
Use cylindrical coordinates to evaluate the below integral.

$$\iiint_R x dV$$

The domain can be drawn as follows:



The projection on x-y plane is



We can convert to cylindrical coordinates.

$x^2+y^2 \le 1 \Rightarrow r: 0 \to 1$

$\theta: 0 \to \frac{\pi}{2}$ [as can be seen in the figure]

$z: 0 \to 2$ [given]

we get

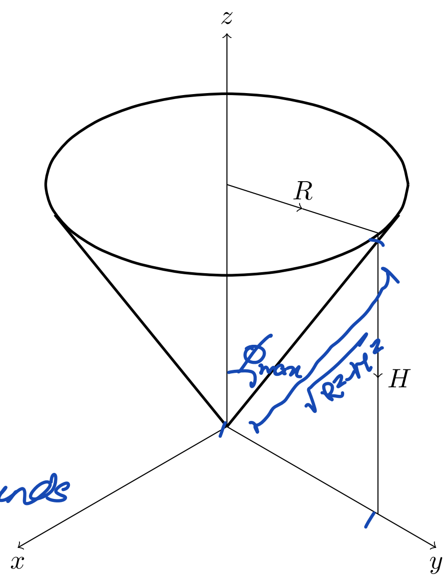
$$\iiint_R x dV = \int_{z=0}^{2}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r\cos\theta \; r dr d\theta dz = \int_{z=0}^{2}\int_{\theta=0}^{\frac{\pi}{2}} \frac{r^3\cos\theta}{3}\Big|_{r=0}^{1} d\theta dz$$

$$= \frac{1}{3}\int_{z=0}^{2}\int_{\theta=0}^{\frac{\pi}{2}} \cos\theta \, d\theta dz = \frac{1}{3}\int_{z=0}^{2} \sin\theta \Big|_{\theta=0}^{\frac{\pi}{2}} dz$$

$$= \frac{1}{3}\int_{z=0}^{2} dz = \frac{2}{3}$$

Problem 10 **10 pts**

Use spherical coordinates to figure out the volume of the below cone.



The upper bound for ρ is given by $z=H$,

i.e. $\rho\cos\phi \le H$

$\rho \le \dfrac{H}{\cos\phi}$

Let us now calculate bounds for ϕ.

$\tan\phi_{max} = \dfrac{R}{H}$, $\phi_{max} = \cos^{-1}\dfrac{H}{\sqrt{R^2+H^2}}$

So $\phi : 0 \to \cos^{-1}\dfrac{H}{\sqrt{R^2+H^2}}$

For θ, we see $\theta : 0 \to 2\pi$ as projection on x-y plane will be a circle.

Now, computing the integral

$$\iiint_K 1\,dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\cos^{-1}\frac{H}{\sqrt{R^2+H^2}}}\int_{\rho=0}^{\frac{H}{\cos\phi}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\cos^{-1}\frac{H}{\sqrt{R^2+H^2}}} \frac{\rho^3\sin\phi}{3}\Big|_0^{\frac{H}{\cos\phi}} d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\cos^{-1}\frac{H}{\sqrt{R^2+H^2}}} \frac{H^3\sin\phi}{3\cos^3\phi}\, d\phi\, d\theta$$

Let $\cos\phi = t$

Then $dt = -\sin\phi\, d\phi$

$\phi = 0 \Rightarrow t = 1$

$\phi = \cos^{-1}\dfrac{H}{\sqrt{R^2+H^2}} \Rightarrow t = \dfrac{H}{\sqrt{R^2+H^2}}$

$$= \frac{H^3}{3}\int_{\theta=0}^{2\pi}\int_{t=1}^{\frac{H}{\sqrt{R^2+H^2}}} \frac{-dt\, d\theta}{t^3}$$

$$= \frac{H^3}{3}\int_0^{2\pi} \frac{1}{2t^2}\Big|_{t=1}^{\frac{H}{\sqrt{R^2+H^2}}} d\theta = \frac{H^3}{3}\int_0^{2\pi}\left(\frac{R^2+H^2}{2H^2} - \frac{1}{2}\right) d\theta = \frac{H^3}{3}\left(\frac{R^2+H^2}{2H^2} - \frac{1}{2}\right)(\theta)_0^{2\pi} = \frac{2\pi H^3}{3}\left(\frac{R^2+H^2}{2H^2} - \frac{1}{2}\right)$$

$$= \frac{\pi R^2 H + \pi H^3}{3} - \frac{\pi H^3}{3} = \frac{\pi R^2 H}{3}$$