Score: $\frac{100}{100}$

Course 32 B Sec. 2

UCLA Department of Mathematics

Fall 2020

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Student:

Student ID:

Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Pr 7	Pr 8	Pr 9	Pr 10	Total
$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{10}$	$\overline{100}$

Midterm 1

Please print your name and student ID in the designated space at the top of the page. Show your work! Answers unsupported by work yield no credit.

Problem 1 Use the symmetry of the problem to find the integral **10 pts**

$$\iint\limits_D \left(1 + \frac{y}{x^{10} + y^{10} + 10}\right) dA$$

over the domain $D = [-3, 2] \times [-1, 1]$.

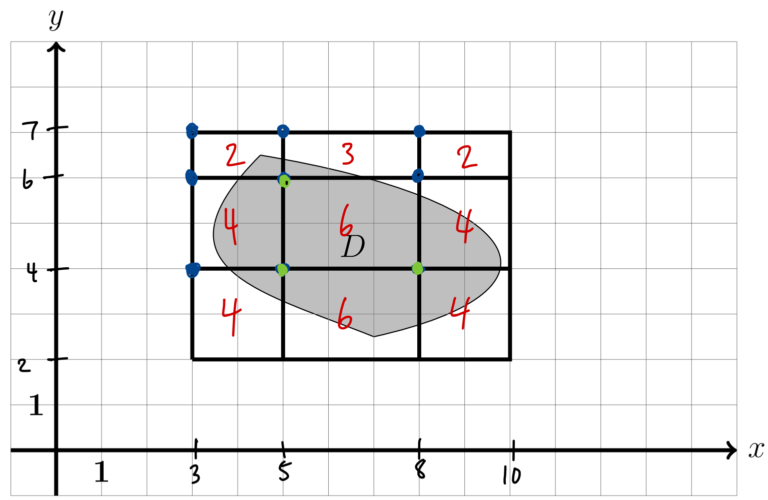
$$\iint\limits_D \left(1 + \frac{y}{x^{10} + y^{10} + 10}\right) dA =$$

Problem 2 **10 pts**

- Use the upper-left vertices of the below partition to find the Riemann sum $S_{3,3}$ for the integral

$$\iint_D (2x-y)dA$$

over the domain D shaded on the picture below. **8 pts**



only 3 points lie in the domain so we only need to account for them

$S_{3,3} =$ 108

$$S_{3,3} = \sum_{i=1}^{N}\sum_{j=1}^{M} \tilde{f}(P_{i,j})\,\Delta X_i\,\Delta Y_j = [6f(5,4)+4f(8,4)+6f(5,6)]$$

$$= [6(2(5)-4)+4(2(8)-4)+6(2(5)-6)] = [6(6)+4(12)+6(4)]$$

$$= 108$$

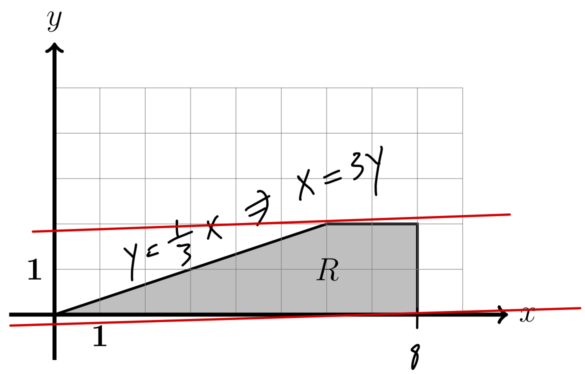
- What is the maximal length $||P||$ of the partition? **2 pts**

$||P|| = 3$

Problem 3 **10 pts**

Should we consider the below region R as vertically simple or as horizontally simple? Please circle the correct answer. **2 pts**

vertically simple   horizontally simple



$0 \le y \le 2$

$3y \le x \le 8$

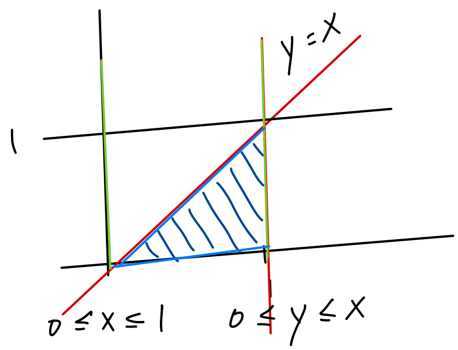
Find the following integral. $\iint_R 8xy\,dA =$ 368 **8 pts**

$$\int_0^2 \int_{3y}^8 8xy\,dx\,dy = \int_0^2 \left[4x^2y\right]_{3y}^8 dy = \int_0^2 256y - 36y^3 dy$$

$$= \left[128y^2 - 9y^4\right]_0^2 = 128(4) - 9(16) - 0 = 368$$

Problem 4 **10 pts**

Evaluate the following integral. Hint: it helps to sketch the domain.



$$\int_0^1 \int_y^1 \frac{\cos x}{x}\, dx dy$$

$0 \le x \le 1 \quad 0 \le y \le x$

$$\int_0^1 \int_y^1 \frac{\cos x}{x} dx dy = \int_0^1 \int_0^x \frac{\cos x}{x} dy dx$$

$$= \int_0^1 \frac{\cos x}{x} \left[y \right]_0^x dx = \int_0^1 \frac{\cos x}{x} x \, dx$$

$$= \int_0^1 \cos x \, dx = \left[\sin x \right]_0^1 = \sin(1) - \sin(0) = \boxed{\sin(1)}$$

Problem 5 **10 pts**

Find the integral

$$\iiint_B 24xy^2z^3 dV$$

over the box $B = [0, a] \times [0, b] \times [0, c]$.

$$\int_0^a \int_0^b \int_0^c 24xy^2z^3 \, dz\, dy\, dx$$

$$= \int_0^a \int_0^b 24xy^2 \left[\frac{1}{4}z^4\right]_0^c dy\, dx$$

$$= \int_0^a \int_0^b 6xy^2c^4 \, dy\, dx$$

$$= \int_0^a 6xc^4 \left[\frac{1}{3}y^3\right]_0^b dx$$

$$= \int_0^a 2xb^3c^4 \, dx$$

$$= b^3c^4\left[x^2\right]_0^a = \boxed{a^2b^3c^4}$$

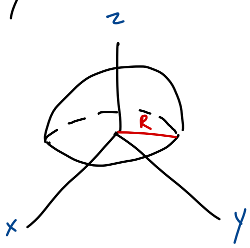
Problem 6 **10 pts**

Compute the average value $\bar{f}$ of the function

$$f(x, y, z) = z$$

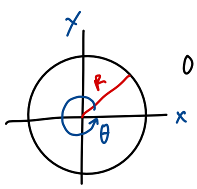
over the region bounded above by the upper semi-sphere of radius R centered at the origin and bounded below by the plane $z = 0$.

$* x^2+y^2=r^2$



$0 \le z \le \sqrt{R^2-x^2-y^2}$

$0 \le z \le \sqrt{R^2-r^2}$



$0 \le \theta \le 2\pi$

$0 \le r \le R$

$$\bar{f} = \frac{\iiint_D f(x,y,z)\,dV}{\iiint_D 1\,dV} = \frac{\int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-r^2}} zr\,dz\,dr\,d\theta}{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{R^2-r^2}} r\,dz\,dr\,d\theta}$$

$$= \frac{\int_0^{2\pi}\int_0^R r\left[\frac{1}{2}z^2\right]_0^{\sqrt{R^2-r^2}} dr\,d\theta}{\int_0^{2\pi}\int_0^R r\left[z\right]_0^{\sqrt{R^2-r^2}} dr\,d\theta} = \frac{\int_0^{2\pi}\int_0^R \frac{1}{2}rR^2 - \frac{1}{2}r^3\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\sqrt{R^2-r^2}\,dr\,d\theta}$$

$u = R^2-r^2$
$du = -2r$

$$= \frac{\frac{1}{2}\int_0^{2\pi}\left[\frac{1}{2}r^2R^2 - \frac{1}{4}r^4\right]_0^R d\theta}{-\frac{1}{2}\int_0^{2\pi}\left[\frac{2}{3}u^{3/2}\right]_{R^2}^0 d\theta} = \frac{\frac{1}{2}\int_0^{2\pi} \frac{1}{2}R^4 - \frac{1}{4}R^4\,d\theta}{\frac{1}{2}\int_0^{2\pi}\frac{2}{3}R^3\,d\theta}$$

$$= \frac{\frac{1}{2}\int_0^{2\pi}\frac{1}{4}R^4\,d\theta}{\frac{1}{3}\int_0^{2\pi}R^3\,d\theta} = \frac{\frac{1}{8}R^4\left[\theta\right]_0^{2\pi}}{\frac{1}{3}R^3\left[\theta\right]_0^{2\pi}} = \frac{\frac{1}{8}R^4\cancel{(2\pi)}}{\frac{1}{3}R^3\cancel{(2\pi)}} = \frac{\frac{1}{8}R}{\frac{1}{3}}$$

$$\boxed{\bar{f} = \frac{3}{8}R}$$

Problem 7 **10 pts**

Convert the following equation
to an equation in rectangular coordinates. **5 pts**

$$r = 2a\sin\theta$$

$r^2 = x^2 + y^2$

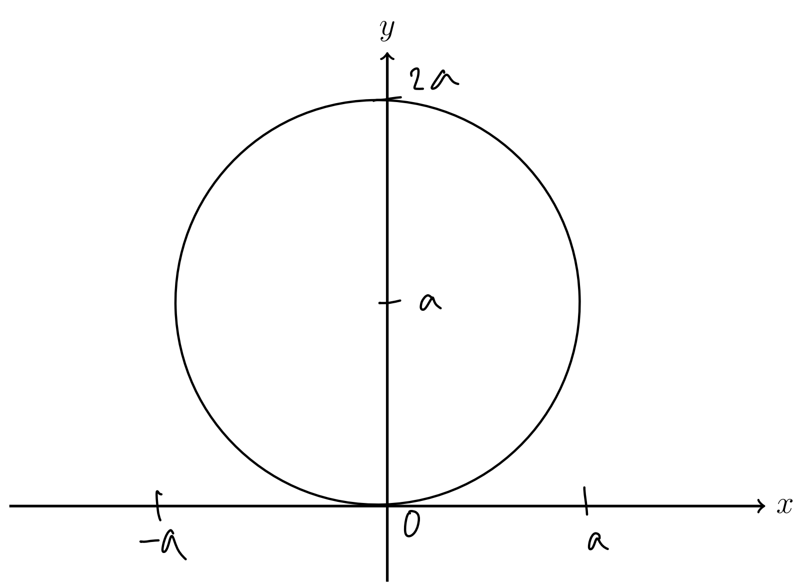
$r\sin\theta = y \Rightarrow \sin\theta = \frac{y}{r}$

$r = 2a\frac{y}{r} \Rightarrow r^2 = 2ay$

$\Rightarrow x^2 + y^2 = 2ay \Rightarrow x^2 + y^2 - 2ay = 0 \Rightarrow x^2 + y^2 - 2ay + a^2 = a^2$

$\boxed{x^2 + (y-a)^2 = a^2}$ → it's a circle of radius 'a' shifted up from the origin by 'a'

Graph the set of the points $(\theta, r(\theta))$ that satisfy the above equation. **5 pts**



Problem 8 **10 pts**

The region of integration R is given by the following inequalities.

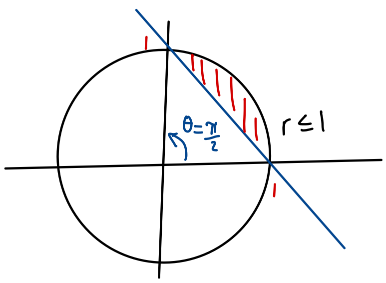
$$R: \quad x^2 + y^2 \leq 1, \quad x + y \geq 1$$

Evaluate the integral below by switching to polar coordinates.

$y \geq 1 - x$

$$\iint_R (x - y)\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$



$r\sin\theta \geq 1 - r\cos\theta$

$r(\sin\theta + \cos\theta) \geq 1$

$\frac{1}{\sin\theta + \cos\theta} \leq r \leq 1$

$0 \leq \theta \leq \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\sin\theta+\cos\theta}}^{1} (r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\sin\theta+\cos\theta}}^{1} r^2(\cos\theta - \sin\theta)\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\cos\theta - \sin\theta) \left[\frac{1}{3} r^3\right]_{\frac{1}{\sin\theta+\cos\theta}}^{1} d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \cos\theta - \sin\theta - \frac{\cos\theta - \sin\theta}{(\sin\theta + \cos\theta)^3}\, d\theta$$

$u = \sin\theta + \cos\theta$
$du = \cos\theta - \sin\theta\, d\theta$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \cos\theta - \sin\theta\, d\theta + \frac{1}{3}\int_{\sin 0 + \cos 0}^{\sin\frac{\pi}{2} + \cos\frac{\pi}{2}} u^{-3}\, du$$

$$= \frac{1}{3}\left[-\sin\theta - \cos\theta\right]_0^{\frac{\pi}{2}} + \frac{1}{3}\left[\frac{1}{-2}u^{-2}\right]_1^1$$

$$= \frac{1}{3}\left(-\sin\frac{\pi}{2} - 0 + 0 + \cos 0\right) - \frac{1}{6}(1 - 1) = 0 - 0 = \boxed{0}$$

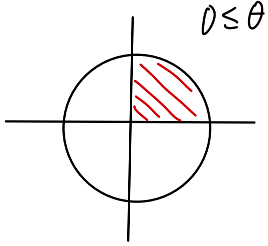
Problem 9 **10 pts**

The region of integration R is given by the following inequalities.

$$R: \quad x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 2$$

Use cylindrical coordinates to evaluate the below integral.

$$\iiint_R x\,dV$$



$0 \leq \theta \leq \frac{\pi}{2}$

$x^2+y^2 \leq 1$

$r^2 \leq 1$

$0 \leq r \leq 1$

$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^2 r \cdot r \cos\theta \, dz\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cos\theta \left[z \right]_0^2 dr\, d\theta$$

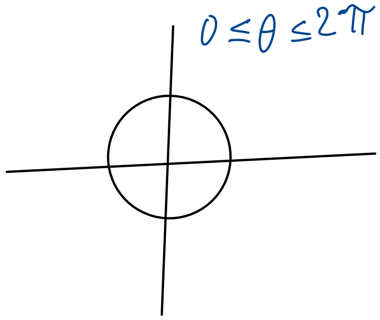
$$= \int_0^{\frac{\pi}{2}} \int_0^1 2r^2 \cos\theta \, dr\, d\theta = \int_0^{\frac{\pi}{2}} 2\cos\theta \left[\frac{1}{3} r^3 \right]_0^1 d\theta$$

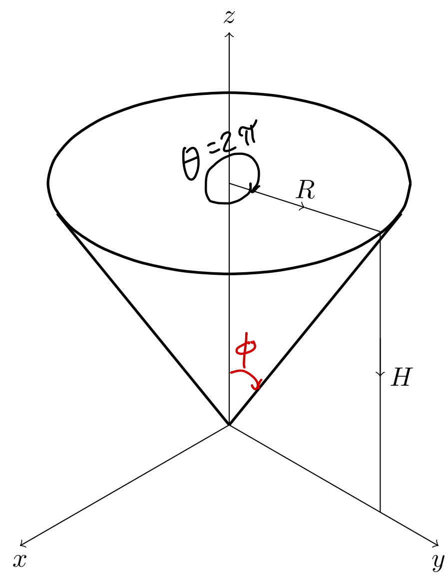
$$= \frac{2}{3} \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \frac{2}{3} \left[\sin\theta \right]_0^{\frac{\pi}{2}} = \frac{2}{3}\left(\sin\frac{\pi}{2} - \sin 0\right) = \boxed{\frac{2}{3}}$$

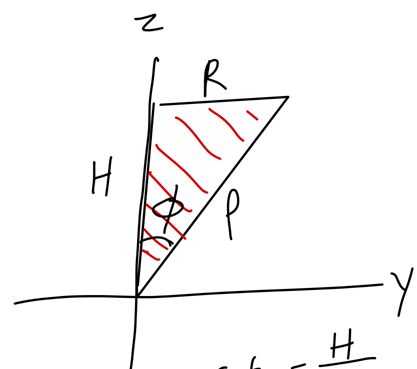
Problem 10 **10 pts**

Use spherical coordinates to figure out the volume of the below cone.

$0 \le \theta \le 2\pi$







$\cos\phi = \frac{H}{\rho}$

$\rho = H\sec\phi$

$0 \le \rho \le H\sec\phi$

cones volume changes by $\frac{H}{R}$ so:

$$z^2 = \frac{H^2}{R^2}(x^2+y^2)$$

$$\rho^2\cos^2\phi = \frac{H^2}{R^2}(\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)$$

$$\rho^2\cos^2\phi = \frac{H^2}{R^2}\rho^2\sin^2\phi \Rightarrow \cos\phi = \frac{H}{R}\sin\phi \Rightarrow \phi = \tan^{-1}\left(\frac{R}{H}\right)$$

$0 \le \phi \le \tan^{-1}\left(\frac{R}{H}\right)$

$$\int_0^{2\pi}\int_0^{\tan^{-1}(\frac{R}{H})}\int_0^{H\sec\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\tan^{-1}(\frac{R}{H})} \sin\phi \left[\frac{1}{3}\rho^3\right]_0^{H\sec\phi} d\phi \, d\theta$$

$u = \tan\phi$
$du = \sec^2\phi$

$$= \int_0^{2\pi}\int_0^{\tan^{-1}(\frac{R}{H})} \frac{H^3\sec^3\phi}{3}\sin\phi \, d\phi \, d\theta = \frac{H^3}{3}\int_0^{2\pi}\int_0^{\tan^{-1}(\frac{R}{H})} \sec^2\phi \frac{\sin\phi}{\cos\phi} d\phi \, d\theta = \frac{H^3}{3}\int_0^{2\pi}\int_0^{\tan^{-1}(\frac{R}{H})} \sec^2\phi\tan\phi \, d\phi \, d\theta$$

$$= \frac{H^3}{3}\int_0^{2\pi}\int_{\tan(0)}^{\tan(\tan^{-1}(\frac{R}{H}))} u \, du \, d\theta = \frac{H^3}{3}\int_0^{2\pi}\left[\frac{1}{2}u^2\right]_0^{\frac{R}{H}} d\theta = \frac{H^3}{6}\int_0^{2\pi}\frac{R^2}{H^2}d\theta = \frac{HR^2}{6}\left[\theta\right]_0^{2\pi} = \frac{HR^2(2\pi)}{6}$$

$$= \boxed{\frac{\pi HR^2}{3}}$$