Course 32 B Sec. 2 UCLA Department of Mathematics

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Pr 1	Pr 2	Pr 3	Pr 4	Pr 5	Pr 6	Pr 7	Pr 8	Pr 9	Pr 10	Total
/10	/10	/10	/10	/10	/10	/10	/10	/10	/10	/100

Midterm 1

Please print your name and student ID in the designated space at the top of the page. Show your work! Answers unsupported by work yield no credit.

Problem 1 Use the symmetry of the problem to find the integral **10 pts**

$$\iint_D \left(1 + \frac{y}{x^{10} + y^{10} + 10}\right) dA$$

over the domain $D = [-3, 2] \times [-1, 1]$.

$$\iint_D \left(1 + \frac{y}{x^{10} + y^{10} + 10}\right) dA = \int_{-3}^{2}\int_{-1}^{1} 1 \, dydx + \int_{-3}^{2}\int_{-1}^{1} \frac{y}{x^{10}+y^{10}+10} \, dydx$$

$\int_{-1}^{1} 1 \, dy = y\Big|_{-1}^{1} = 2$

$\int_{-1}^{1} \frac{y}{x^{10}+y^{10}+10} \, dy = 0$ (symmetry, odd function)

$\int_{-3}^{2} 2 \, dx = 2x\Big|_{-3}^{2} = 10$

$\int_{-3}^{2} 0 \, dx = 0$

$10 + 0 = \boxed{10}$

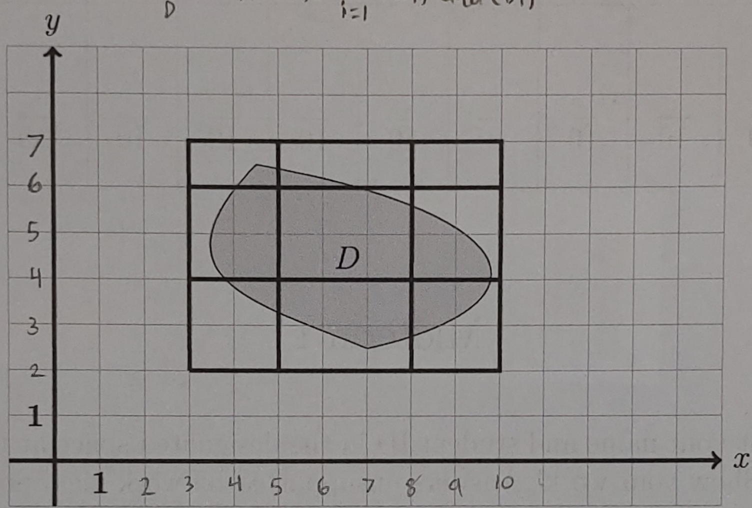
Problem 2 **10 pts**

- Use the upper-left vertices of the below partition to find the Riemann sum $S_{3,3}$ for the integral

$$\iint_D (2x - y)dA$$

over the domain D shaded on the picture below. **8 pts**

$$\iint_D f(x,y)\, dA \approx \sum_{i=1}^{n} f(P_i)\ \text{area}\ (D_i)$$



$$\begin{array}{r} 48 \\ +36 \\ \hline 84 \\ +64 \\ \hline 148 \\ +6 \\ \hline 154 \\ +27 \\ \hline 181 \end{array}$$

$S_{3,3} = [2(3)-(7)](2)(1) + [2(5)-(7)](3)(1) + [2(8)-(7)](2)(1) + [2(3)-(6)](2)(2) + [2(5)-(6)](3)(2)$
$+ [2(8)-(6)](2)(2) + [2(3)-(4)](2)(2) + [2(5)-(4)](3)(2) + [2(8)-(4)](2)(2)$

$S_{3,3} = (-1)(2) + (3)(3) + (9)(2) + (0)(4) + (4)(6) + (10)(4) + (2)(4) + (6)(6) + (12)(4)$

$S_{3,3} = -2 + 9 + 18 + 0 + 24 + 40 + 8 + 36 + 48$

$\boxed{S_{3,3} = 181}$

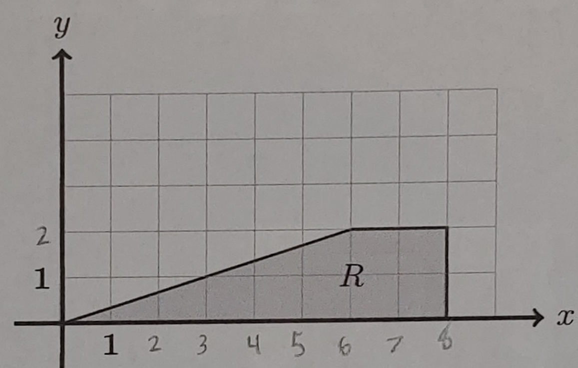
- What is the maximal length $||P||$ of the partition? **2 pts**

$||P|| = \boxed{3}$

Problem 3 **10 pts**

Should we consider the below region R as vertically simple or as horizontally simple? Please circle the correct answer. **2 pts**

vertically simple horizontally simple



$y = \frac{1}{3}x$

$x = 3y$

Find the following integral. $\iint_R 8xy\,dA = \int_0^2 \int_{3y}^8 8xy\,dx\,dy$ **8 pts**

$\int_{3y}^{8} 8xy\,dx = 8y(\frac{1}{2}x^2)\Big|_{3y}^{8} = 4y[64 - 9y^2] = 256y - 36y^3$

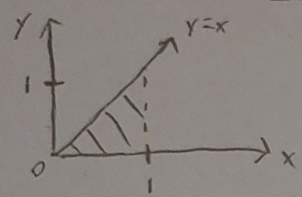
$\int_0^2 (256y - 36y^3)\,dy = 128y^2\Big|_0^2 - 9y^4\Big|_0^2 = 128(4) - 9(16) = 512 - 144 = \boxed{368}$

$16 \times 9 = 144$

$512 - 144 = 368$

Problem 4 **10 pts**

Evaluate the following integral. Hint: it helps to sketch the domain.



$$\int_0^1 \int_y^1 \frac{\cos x}{x}\, dx dy$$

$\int_0^1 \int_y^1 \frac{\cos x}{x}\, dx\, dy = \int_0^1 \int_0^x \frac{\cos x}{x}\, dy\, dx$

$\int_0^x \frac{\cos x}{x}\, dy = \frac{\cos x}{x} (y)\Big|_0^x = \cos x$

$\int_0^1 \cos x\, dx = \sin x \Big|_0^1 = \boxed{\sin 1}$

Problem 5 **10 pts**

Find the integral

$$\iiint_B 24xy^2z^3 dV$$

over the box $B = [0,a] \times [0,b] \times [0,c]$.

$\iiint_B 24xy^2z^3\, dV = \int_0^a \int_0^b \int_0^c 24xy^2z^3\, dz\,dy\,dx$

$\int_0^c 24xy^2z^3\, dz = 24xy^2\left(\frac{1}{4}z^4\right)\Big|_0^c = 6c^4xy^2$

$\int_0^b 6c^4xy^2\, dy = 6c^4x\left(\frac{1}{3}y^3\right)\Big|_0^b = 2b^3c^4x$

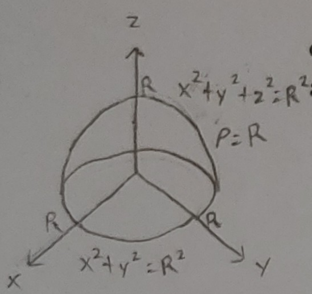
$\int_0^a 2b^3c^4x\, dx = 2b^3c^4\left(\frac{1}{2}x^2\right)\Big|_0^a = \boxed{a^2b^3c^4}$

Problem 6 **10 pts**

Compute the average value $\bar{f}$ of the function

$$f(x, y, z) = z$$

over the region bounded above by the upper semi-sphere of radius R centered at the origin and bounded below by the plane $z = 0$.



$$\bar{f} = \frac{1}{vol(D)} \iiint_D z \, dV = \frac{1}{\frac{1}{2}\left(\frac{4}{3}\pi R^3\right)} \int_0^{\pi/2} \int_0^{2\pi} \int_0^R \rho\cos\phi \, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\int_0^R \rho^3 \sin\phi \cos\phi \, d\rho = \sin\phi\cos\phi\left(\frac{1}{4}\rho^4\right)\Big|_0^R = \frac{1}{4}R^4 \sin\phi\cos\phi$$

$$\int_0^{2\pi} \frac{1}{4}R^4 \sin\phi\cos\phi \, d\theta = \frac{1}{4}R^4 \sin\phi\cos\phi \, (\theta)\Big|_0^{2\pi} = \frac{\pi}{2}R^4 \sin\phi\cos\phi$$

$$\int_0^{\pi/2} \frac{\pi}{2}R^4 \sin\phi\cos\phi \, d\phi = \frac{\pi}{2}R^4 \int_0^{\pi/2} \frac{\sin 2x}{2} \, dx = \frac{\pi}{4}R^4\left(-\frac{1}{2}\cos 2x\right)\Big|_0^{\pi/2} = \frac{\pi}{4}R^4\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{\pi}{4}R^4$$

$$\bar{f} = \frac{3}{2\pi R^3}\left(\frac{\pi}{4}R^4\right) = \boxed{\frac{3}{8}R}$$

Problem 7

10 pts

Convert the following equation to an equation in rectangular coordinates. **5 pts**

$r=\sqrt{x^2+y^2}$ $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

$$r = 2a\sin\theta$$ $x=r\cos\theta$ $y=r\sin\theta$

$r^2=2ar\sin\theta$

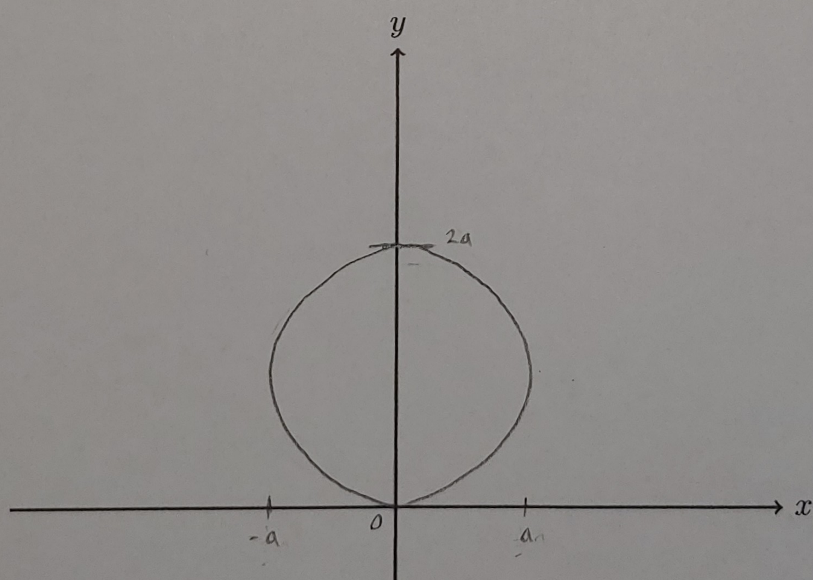
$x^2+y^2=2ay$

$x^2+y^2-2ay=0$

$x^2+(y^2-2ay+a^2)=a^2$

$\boxed{x^2+(y-a)^2=a^2}$

Graph the set of the points $(\theta, r(\theta))$ that satisfy the above equation. **5 pts**



Problem 8 **10 pts**

The region of integration R is given by the following inequalities.

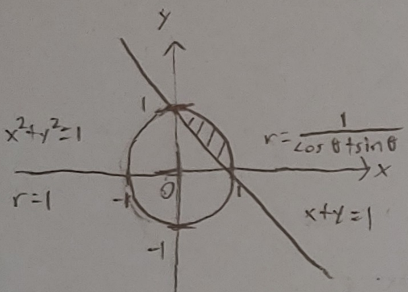
$$R:\ x^2+y^2 \le 1,\quad x+y \ge 1$$

$r\cos\theta + r\sin\theta = 1$

$r = \frac{1}{\cos\theta + \sin\theta}$

Evaluate the integral below by switching to polar coordinates.

$$\iint_R (x-y)dA$$



$\iint_R (x-y)\,dA = \int_0^{\pi/2}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} (r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$

$\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^2(\cos\theta-\sin\theta)\,dr = (\cos\theta-\sin\theta)\frac{1}{3}r^3\Big|_{\frac{1}{\cos\theta+\sin\theta}}^{1} = \frac{1}{3}(\cos\theta-\sin\theta)\left[1-\frac{1}{(\cos\theta+\sin\theta)^3}\right]$

$\int_0^{\pi/2}\frac{1}{3}(\cos\theta-\sin\theta)\left(1-\frac{1}{(\cos\theta+\sin\theta)^3}\right)d\theta = \int_1^1 \frac{1}{3}\left(1-\frac{1}{w^3}\right)dw = \boxed{0}$

Let $w = \cos\theta + \sin\theta$

$dw = -\sin\theta + \cos\theta\, d\theta$

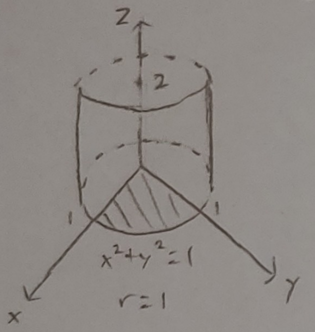
Problem 9 **10 pts**

The region of integration R is given by the following inequalities.

$$R: \ x^2 + y^2 \leq 1, \ \ x \geq 0, \ \ y \geq 0, \ \ 0 \leq z \leq 2$$

Use cylindrical coordinates to evaluate the below integral.

$$\iiint_R x dV$$



$$\iiint_R x \, dV = \int_0^2 \int_0^{\pi/2} \int_0^1 r\cos\theta \, r dr d\theta dz$$

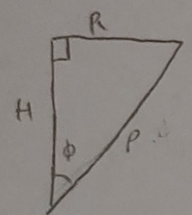
$$\int_0^1 r^2 \cos\theta \, dr = \cos\theta \left(\tfrac{1}{3} r^3\right)\Big|_0^1 = \tfrac{1}{3}\cos\theta$$

$$\int_0^{\pi/2} \tfrac{1}{3}\cos\theta \, d\theta = \tfrac{1}{3}\sin\theta \Big|_0^{\pi/2} = \tfrac{1}{3}$$

$$\int_0^2 \tfrac{1}{3} \, dz = \tfrac{1}{3} z \Big|_0^2 = \boxed{\tfrac{2}{3}}$$

Problem 10 **10 pts**

Use spherical coordinates to figure out the volume of the below cone.



$\tan \phi = \frac{R}{H}$

$\phi = \tan^{-1}\left(\frac{R}{H}\right)$

$0 \le \phi \le \tan^{-1}\left(\frac{R}{H}\right)$

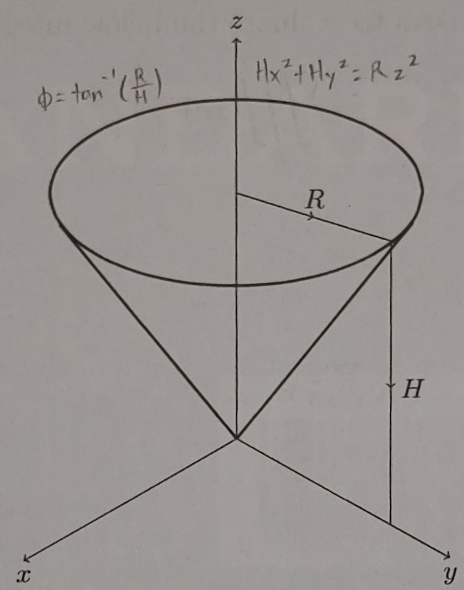
$\cos \phi = \frac{H}{\rho}$

$\rho = \frac{H}{\cos \phi}$

$0 \le \rho \le \frac{H}{\cos \phi}$

$\phi = \tan^{-1}\left(\frac{R}{H}\right)$

$Hx^2 + Hy^2 = Rz^2$



$$V = \iiint_R dV = \int_0^{\tan^{-1}\left(\frac{R}{H}\right)} \int_0^{2\pi} \int_0^{\frac{H}{\cos\phi}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\rho$$

$$\int_0^{\frac{H}{\cos\phi}} \rho^2 \sin\phi \, d\rho = \sin\phi \left(\tfrac{1}{3}\rho^3\right)\Big|_0^{\frac{H}{\cos\phi}} = \tfrac{1}{3} H^3 \tan\phi \sec^2\phi$$

$$\int_0^{2\pi} \tfrac{1}{3} H^3 \tan\phi \sec^2\phi \, d\theta = \tfrac{1}{3} H^3 \tan\phi \sec^2\phi \, (\theta)\Big|_0^{2\pi} = \tfrac{2\pi}{3} H^3 \tan\phi \sec^2\phi$$

$$\int_0^{\tan^{-1}\left(\frac{R}{H}\right)} \tfrac{2\pi}{3} H^3 \tan\phi \sec^2\phi \, d\phi = \tfrac{2\pi}{3} H^3 \int_0^{\frac{R}{H}} w \, dw = \tfrac{2\pi}{3} H^3 \left(\tfrac{1}{2} w^2\right)\Big|_0^{\frac{R}{H}} = \tfrac{\pi}{3} H^3 \left(\tfrac{R^2}{H^2}\right) = \boxed{\tfrac{1}{3}\pi R^2 H}$$

Let $w = \tan\phi$

$dw = \sec^2\phi \, d\phi$