Course 32 B

Spring 2021

UCLA Department of Mathematics

Instructor: Oleg Gleizer

# Final Exam

Please print your name and student ID in the designated space below. Show your work! Answers unsupported by work yield no credit.

Student's Name, First:____________________ Last:_______________________

Student ID:______________________________

**Pledge:** I assert, on my honor, that I have not received assistance of any kind from any other person while working on the final and that I have not used any non-permitted materials or technologies during the period of this evaluation.

**Student's signature:** ______________________________________________

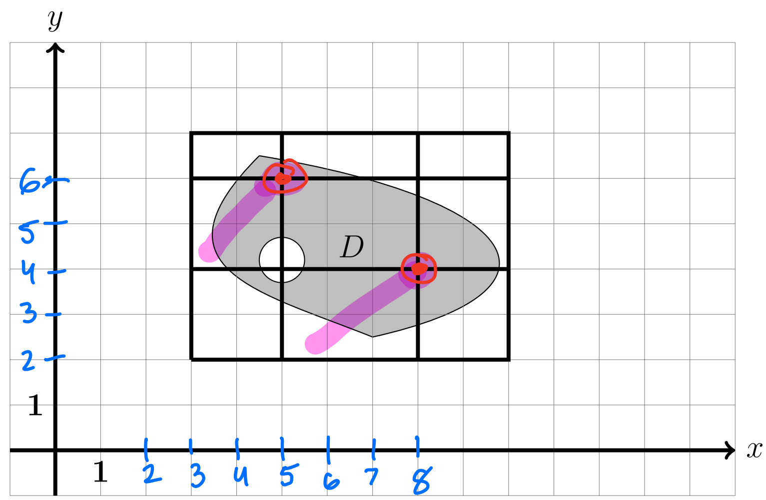
| Pr 1 | Pr 2 | Pr 3 | Pr 4 | Pr 5 | Pr 6 | Pr 7 | Pr 8 | Total |
|---|---|---|---|---|---|---|---|---|
| $\overline{10}$ | $\overline{10}$ | $\overline{10}$ | $\overline{10}$ | $\overline{10}$ | $\overline{10}$ | $\overline{25}$ | $\overline{15}$ | $\overline{100}$ |

**Problem 1** **10 pts**

- Use the upper-right vertices of the below partition to find the Riemann sum $S_{3,3}$ for the integral

$$\iint_D (x^2 - y^2)dA$$

over the domain $D$ shaded on the picture below. **8 pts**




$S_{3,3} =$ $4 * f(5,6) + 6 * f(8,4)$

$4*(25-36) + 6*(64-16)$

$244$

- What is the maximal length $||P||$ of the partition? **2 pts**

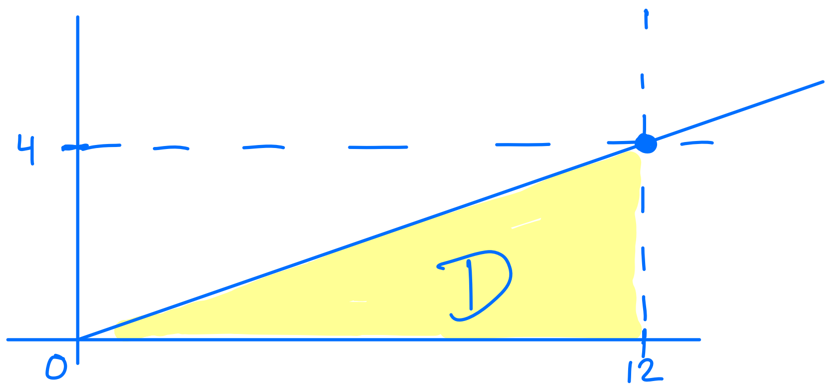
$||P|| =$ $3$

**Problem 2** **10 pts**

Consider the following integral.

$$\int_0^4 \int_{3y}^{12} e^{x^2}\, dx\, dy$$

- Sketch the domain of integration. **2 pts**




$3y = x$

$y = \frac{x}{3}$

- Switch the order of integration and evaluate. **8 pts**

$0 \le x \le 12$

$0 \le y \le \frac{x}{3}$

$$\int_0^{12} \int_0^{x/3} e^{x^2}\, dy\, dx \qquad e^{x^2} y \Big]_{\emptyset}^{x/3} \qquad \frac{1}{3} \int_0^{12} e^{x^2} \cdot x \, dx$$

$u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

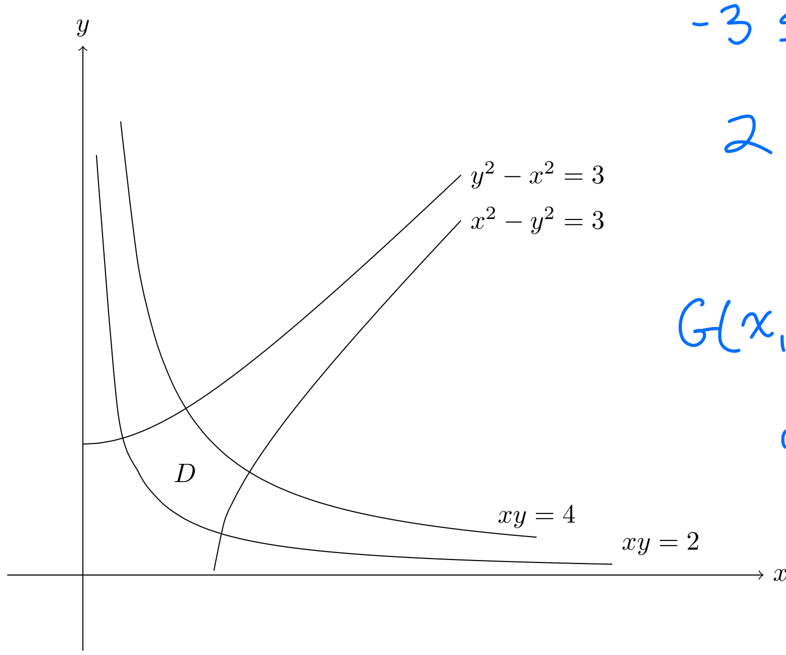
$$\frac{1}{6} \int_0^{144} e^u \, du \qquad \frac{1}{6} e^u \Big]_0^{144}$$

$$\frac{1}{6}\left(e^{144} - 1\right)$$

**Problem 3** **10 pts**

$D$ is the region bounded by the curves on the picture below.
Find the following integral.

$$\iint_D (x^2 + y^2)\,dA$$




$$-3 \le \overset{u}{x^2 - y^2} \le 3$$

$$2 \le \overset{v}{xy} \le 4$$

$$G(x,y) \longmapsto (\overset{u}{x^2-y^2}, \overset{v}{xy})$$

$$dudv = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} dxdy$$

$$dudv = \overset{2x^2+2y^2}{2(x^2+y^2)}\,dxdy$$

$$\frac{dudv}{2(x^2+y^2)} = dxdy$$

$$\int_2^4 \int_{-3}^{3} \cancel{(x^2+y^2)} \cdot \frac{dudv}{2\cancel{(x^2+y^2)}}$$

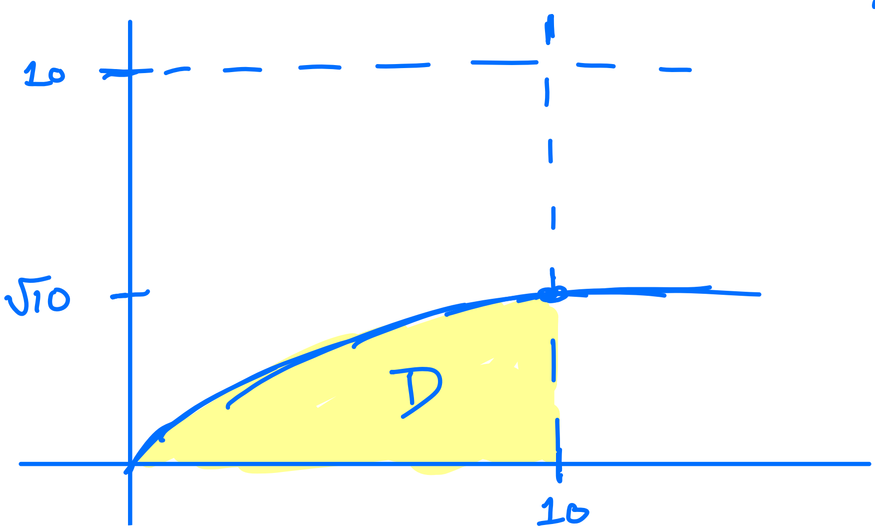
$$\frac{1}{2} * 6 * 2 = 6$$

**Problem 4** **10 pts**

The real numbers $X$ and $Y$ are randomly and independently chosen between zero and ten. The joint probability density is

$$p(x,y) = \begin{cases} 0.01 & \text{if } (x,y) \in [0,10] \times [0,10] \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability $P$ that $X \geq Y^2$.




$y^2 \le x$

$y^2 \le 10$

$y \le \sqrt{10}$

$0 \le x \le 10$

$0 \le y \le \sqrt{x}$

$$\int_0^{10} \int_0^{\sqrt{x}} 0.01 \, dy\, dx$$

$$0.01 \int_0^{10} \sqrt{x} \, dx \qquad \frac{2}{3} \cdot \frac{1}{100} \, x^{3/2} \Big]_0^{10}$$

$$\frac{\overset{1}{\cancel{2}}}{3} \cdot \frac{1}{\underset{\underset{5}{\cancel{10}}}{\cancel{100}}} \cdot \overset{1}{\cancel{10}}\sqrt{10}$$

$$\frac{\sqrt{10}}{15}$$

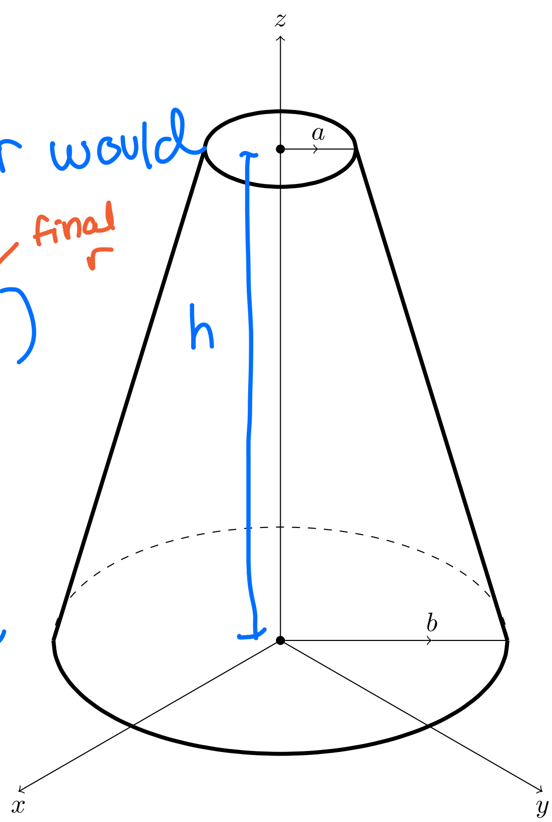
In Problems 5 and 6 of this test, you are asked to consider a truncated straight circular cone $\mathcal{C}$. Its lower base is a circle of radius $b$ located in the $xy$-plane and centered at the origin. Its upper base is a circle of radius $a$ located in the plane $z = h$ and centered at the point $P = (0, 0, h)$ as shown on the picture below. The cone is homogeneous of mass $M$.

For a full cone, r would be:

initial r

final r

$$R - \frac{z}{h}(R - 0)$$




For truncated cone, r should be:

$$b - (b-a)\frac{z}{h}$$

$\frac{z}{h}$ determines the rate @ which the difference between the 2 radii manifests

**Problem 5** **10 pts**

Find the volume of $\mathcal{C}$. The answer without the derivation yields only one point!

$$0 \le \theta \le 2\pi \qquad 0 \le z \le h \qquad \underbrace{0 \le r \le b-(b-a)\frac{z}{h}}_{\text{See above for derivation}}$$

$$\int_0^{2\pi}\int_0^h\int_0^{b-(b-a)\frac{z}{h}} r\,dr\,dz\,d\theta$$

$$2\pi\int_0^h\int_0^{b-(b-a)\frac{z}{h}} r\,dr\,dz$$

$$\pi\int_0^h r^2\Big]_{\emptyset}^{b-(b-a)\frac{z}{h}} dz \qquad \left(b-(b-a)\frac{z}{h}\right)^2 - \emptyset$$

$$\pi\int_0^h b^2 - 2b(b-a)\frac{z}{h} + (b-a)^2\frac{z^2}{h^2}\,dz$$

$$\pi\left( b^2z - \frac{b(b-a)z^2}{h} + \frac{(b-a)^2z^3}{3h^2}\right]_{\emptyset}^{h}$$

$$\pi\left( b^2h - \underset{-b^2h+abh}{(b^2-ab)h} + \frac{(b-a)^2h}{3}\right)$$

$$\frac{(b^2-2ab+a^2)h}{3}$$



$$ab(h) + \frac{b^2(h)}{3} - \frac{2ab}{3}(h) + \frac{a^2}{3}(h)$$

$$\pi\left(\frac{ab(h)}{3} + \frac{b^2}{3}(h) + \frac{a^2}{3}(h)\right) = \frac{\pi h}{3}(ab+a^2+b^2)$$

**Problem 6** **10 pts**

Recall that the cone is homogeneous and has mass $M$. Find the cone's moment of inertia with respect to the $z$-axis, $I_z$.

$$I_z = \iiint_W (x^2+y^2)\,\mu(x,y,z)\,dV$$

$$x = r\cos\theta$$

$$y = r\cos\theta$$

$$(r^2)r$$

$$\mu = C = \frac{M}{V}$$

$$\frac{3M\cdot 2\cancel{\pi}}{\cancel{\pi}h(ab+a^2+b^2)}$$

$$\frac{6M}{h(ab+a^2+b^2)}\int_0^h\int_0^{b-(b-a)\frac{z}{h}} r^3\,dr\,dz$$

$$\left(\frac{bh-(b-a)z}{h}\right)^4$$

$$\frac{3\cancel{6}}{\cancel{4}\,2}$$

$$\frac{3M}{2h(ab+a^2+b^2)}\int_0^h\left(b-(b-a)\frac{z}{h}\right)^4 dz$$

$$\frac{3M}{2h^5(ab+a^2+b^2)}\int_{\emptyset}^h\left(bh-(b-a)z\right)^4 dz$$

$$\frac{(bh-(b-a)z)^5}{-5(b-a)}$$

$$\frac{3M}{10h^5(a-b)(ab+a^2+b^2)}\left( \Big[ (bh-(b-a)z)^5 \Big]_0^h \right.$$

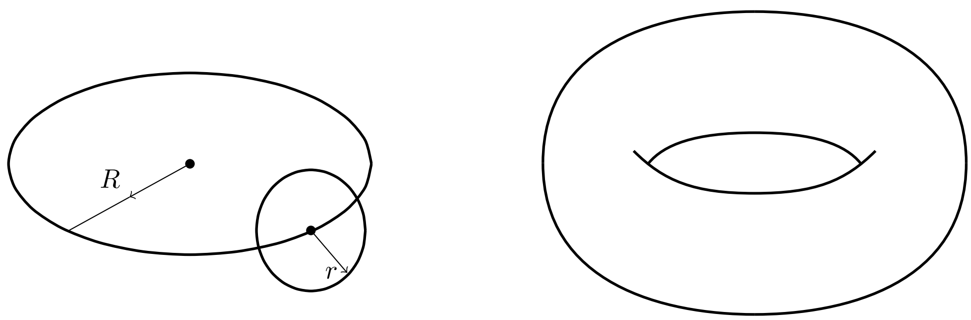
$$\left( (ah)^5-(bh)^5 \right)$$

$$h^5\left(a^5-b^5\right)$$

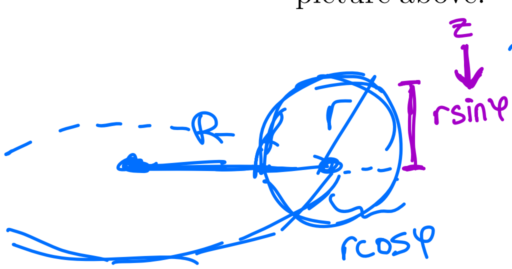
$$I_z = \frac{3M(a^5-b^5)}{10(a-b)(ab+a^2+b^2)}$$

**Problem 7** **25 pts**

A torus $T(R, r)$ is a 2D surface spanned by a circumference of radius $r$, its center rotating around a circumference of radius $R > r$, the plane of the smaller circumference being always perpendicular to the larger circumference.




- Assuming that the larger circumference lies in the $xy$-plane and is centered at the origin, find the parametrization $(x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi))$ of the torus where $0 \le \theta < 2\pi$ is the angle the radius-vector of the larger circumference forms with the $x$-axis and $0 \le \varphi < 2\pi$ is the angle the radius-vector of the smaller circumference forms with the $xy$-plane as shown on the left-hand side picture above. **5 pts**




$x = (R + r\cos\varphi)\cos\theta$

$y = (R + r\cos\varphi)\sin\theta$

$z = r\sin\varphi$

$R' = R + r\cos\varphi$

$x = R'\cos\theta$

$y = R'\sin\theta$

$$G(\theta, \varphi) \longmapsto \langle \overset{x}{(R + r\cos\varphi)\cos\theta}, \overset{y}{(R + r\cos\varphi)\sin\theta}, \overset{z}{r\sin\varphi} \rangle$$

$$\langle R\cos\theta + r\cos\varphi\cos\theta, R\sin\theta + r\cos\varphi\sin\theta, r\sin\varphi \rangle$$

$T_\theta = -R\sin\theta - r\cos\varphi\sin\theta\,,\quad R\cos\theta + r\cos\varphi\cos\theta\,,\quad \emptyset$

$-\sin\theta(R+r\cos\varphi) \qquad \cos\theta(R+r\cos\varphi) \qquad \emptyset$

$T_\varphi = \langle -r\cos\theta\sin\varphi, -r\sin\theta\sin\varphi, r\cos\varphi\rangle$

- Find the vectors $\vec{T}_\theta$ and $\vec{T}_\varphi$. **4 pts**

$T_\theta = \langle -\sin\theta(R+r\cos\varphi), \cos\theta(R+r\cos\varphi), 0\rangle$

$T_\varphi = \langle -r\cos\theta\sin\varphi, -r\sin\theta\sin\varphi, r\cos\varphi\rangle$

- Find the normal vector $\vec{N} = \vec{T}_\theta \times \vec{T}_\varphi$. **6 pts**

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\sin\theta(R+r\cos\varphi) & \cos\theta(R+r\cos\varphi) & \emptyset \\ -r\cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\varphi \end{vmatrix}$$

$r\cos\varphi\cos\theta(R+r\cos\varphi)$

$Rr\cos\varphi\cos\theta + r^2\cos^2\varphi\cos\theta$

$r\sin\theta\cos\varphi(R+r\cos\varphi)$

$Rr\sin\theta\cos\varphi + r^2\sin\theta\cos^2\varphi$

$\vec{N} = \langle Rr\cos\varphi\cos\theta + r^2\cos^2\varphi\cos\theta, Rr\sin\theta\cos\varphi + r^2\sin\theta\cos^2\varphi,$
$Rr\sin\varphi + r^2\sin\varphi\cos\varphi\rangle$

The problem continues to the next page.



$r\sin^2\theta\sin\varphi(R+r\cos\varphi) + r\cos^2\theta\sin\varphi(R+r\cos\varphi)$

$Rr\sin^2\theta\sin\varphi + r^2\sin^2\theta\sin\varphi\cos\varphi + Rr\cos^2\theta\sin\varphi + r^2\cos^2\theta\sin\varphi\cos\varphi$

- Find the length of $\vec{N}$. **5 pts**

$$(Rr\cos\varphi\cos\theta + r^2\cos^2\varphi\cos\theta)^2 + (Rr\sin\theta\cos\varphi + r^2\sin\theta\cos^2\varphi)^2$$
$$+ (Rr\sin\varphi + r^2\sin\varphi\cos\varphi)^2$$

$$\Big(\cancel{R^2r^2\cos^2\varphi\cos^2\theta} + \boxed{\cancel{2Rr^3\cos^3\varphi\cos^2\theta}} + \cancel{r^4\cos^4\varphi\cos^2\theta}$$
$$+ \cancel{R^2r^2\sin^2\theta\cos^2\varphi} + \boxed{\cancel{2Rr^3\sin^2\theta\cos^3\varphi}} + \cancel{r^4\sin^2\theta\cos^4\varphi}$$
$$+ R^2r^2\sin^2\varphi + 2Rr^3\sin^2\varphi\cos\varphi + r^4\sin^2\varphi\cos^2\varphi\Big)$$

$$\cancel{R^2r^2\cos^2\varphi} + 2Rr^3\cos^3\varphi + \cancel{r^4\cos^4\varphi} + \cancel{R^2r^2\sin^2\varphi}$$
$$+ \cancel{2Rr^3\sin^2\varphi\cos\varphi} + \cancel{r^4\sin^2\varphi\cos^2\varphi}$$

$$R^2r^2 + r^4\cos^2\varphi\underbrace{\cancel{(\cos^2\varphi + \sin^2\varphi)}}_{1}$$
$$+ 2Rr^3\cos\varphi\underbrace{\cancel{(\cos^2\varphi + \sin^2\varphi)}}_{1}$$

The problem continues to the next page.



$$\|\vec{N}\| = \sqrt{\overbrace{(Rr + r^2\cos\varphi)^2}^{R^2r^2 + r^4\cos^2\varphi + 2Rr^3\cos\varphi}} = \boxed{Rr + r^2\cos\varphi}$$

- Find the area of the torus. The correct answer unsupported by work yields one point only. **5 pts**

$$\text{Area} = \int_0^{2\pi}\int_0^{2\pi} Rr + r^2\cos\varphi \, d\varphi \, d\theta$$

$$Rr\varphi - r^2 \sin\varphi \Big]_0^{2\pi}$$

$$\cancel{0}$$

$$2\pi Rr$$

$$= 2\pi Rr \int_0^{2\pi} d\theta = 2\pi Rr \big( \theta \big]_0^{2\pi}$$

$$2\pi Rr * 2\pi$$

$$4\pi^2 Rr$$

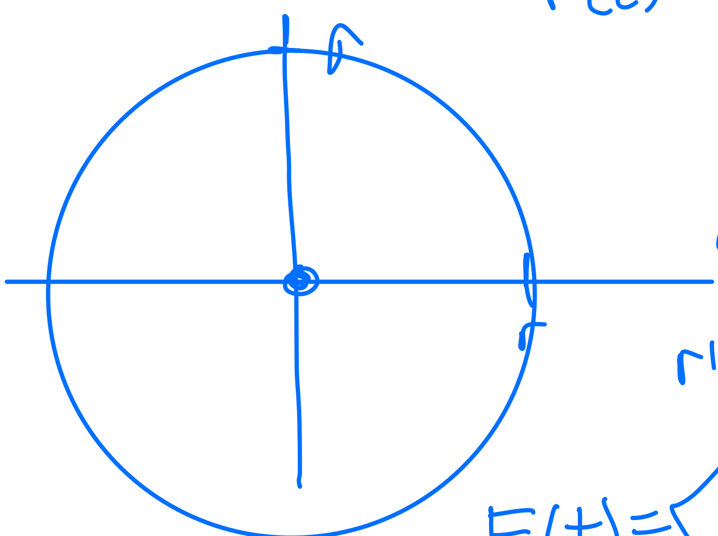
**Problem 8** **15 pts**

- Find the circulation of the vortex field

$$\vec{F} = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

along a positively oriented circumference $C_1$ of radius $r$ centered at the origin. **5 pts**

$r(t) = \langle r\cos t, r\sin t \rangle$ for $0 \le t \le 2\pi$




$$\int_{C_1} \vec{F}(t) \cdot r'(t)\, dt$$

$r'(t) = \langle -r\sin t, r\cos t \rangle$

$$F(t) = \left\langle \frac{-\cancel{r}\sin t}{r^{\cancel{2}}}, \frac{\cancel{r}\cos t}{r^{\cancel{2}}} \right\rangle$$

$\vec{F}(t) \cdot r'(t)$

$\sin^2 t + \cos^2 t = 1$

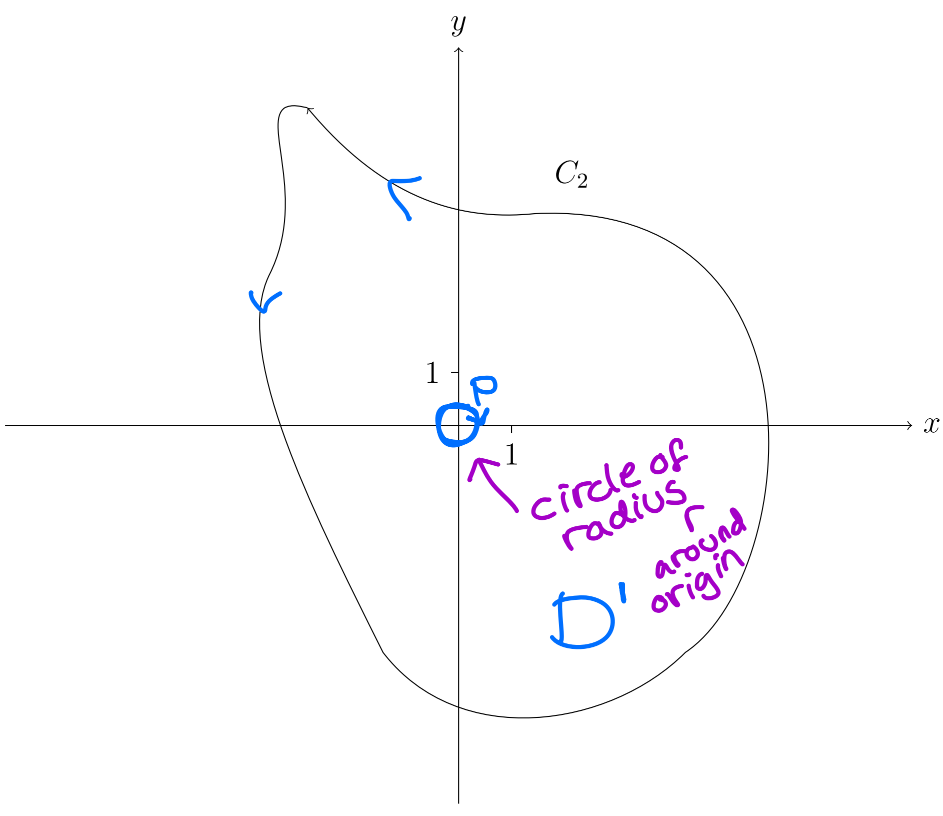
$$\int_{\emptyset}^{2\pi} dt = t \Big]_{\emptyset}^{2\pi}$$

The problem continues to the next page.

$\boxed{2\pi}$

- Let $C_2$ be a smooth, simple, positively oriented path around the origin in the $(x, y)$-plane as on the picture below. Use Green's Theorem to find the following integral. **10 pts**

$$\oint_{C_2} \vec{F} \cdot d\vec{p}$$




* Let's call D' the domain bounded by $C_2 - P$



* Bc D' lacks hole we can use Green's Theorem:

$$\oint_{D'} \vec{F}\cdot d\vec{p} = \iint_{D'} \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{\text{we know this is } \emptyset \text{ for a vortex field}} dx\,dy$$

$$= 0$$

$$\oint_{D'} \vec{F}\cdot d\vec{p} = \emptyset = \oint_{C_2} \vec{F}\cdot d\vec{p} - \underbrace{\oint_{P} \vec{F}\cdot d\vec{p}}_{\text{we know this is } 2\pi \text{ by the 1st part of problem}}$$

$$\emptyset = \oint_{C_2} \vec{F}\cdot d\vec{p} - 2\pi$$

$$\oint_{C_2} \vec{F}\cdot d\vec{p} = \boxed{2\pi}$$