21S-MATH32B-3 Final Exam

RICHARD JIANG

TOTAL POINTS

100 / 100

QUESTION 1

1 Problem 1 10 / 10

✓ **- 0 pts Correct $$S_{3,3} = 244$$ with correct work**

✓ **- 0 pts Correct $$\|P\| = 3$$**

- **2 pts** Areas incorrect
- **4 pts** Did not multiply function values by areas
- **3 pts** Did not take into account shape of integration domain by excluding test points that lie outside of it
- **2 pts** Used evenly spaced points instead of given partition
- **2 pts** Incorrect $$\|P\|$$
- **1 pts** Included test point (5,4) in hole in domain

QUESTION 2

2 Problem 2 10 / 10

✓ **- 0 pts Accurate sketch**

✓ **- 0 pts Correct integral = $$\frac{1}{6}(e^{144} - 1)$$ with correct work**

- **1 pts** Arithmetic errors
- **1 pts** Incorrect bound on new integral
- **1 pts** Triangle backwards in sketch
- **1 pts** Sketched line $$x=3y$$ but not domain of integration
- **2 pts** Incorrect sketch

QUESTION 3

3 Problem 3 10 / 10

✓ **- 0 pts Correct integral (6) with correct work**

- **1.5 pts** Off by minus sign: change of variables factor is absolute value of determinant
- **2.5 pts** Wrong bounds of new integral
- **0.5 pts** One wrong bound on new integral
- **0.5 pts** Arithmetic error
- **8 pts** Incorrect, but some work

QUESTION 4

4 Problem 4 10 / 10

✓ **- 0 pts Correct**

- **2 pts** Misinterpreting the $$X\geq Y^2$$ condition
- **2 pts** Giving an answer that could not be a probability (A negative number or a number more than 1)
- **1 pts** Arithmetic Mistake
- **10 pts** No Answer Given

QUESTION 5

5 Problem 5 10 / 10

✓ **- 0 pts Correct**

- **6 pts** Major mistakes in setting up the integral
- **3 pts** Major mistakes in evaluating the integral
- **6 pts** Major parts of the problem left unsolved

QUESTION 6

6 Problem 6 10 / 10

✓ **- 0 pts Correct**

- **5 pts** Incorrectly setting up the integral, or setting up an integral for the wrong moment of Inertia
- **3 pts** Incorrectly reading the instructions
- **1 pts** Arithmetic Mistake
- **3 pts** Failing to incorporate the density in the final answer
- **3 pts** Mistakes in the Evaluation of the integral
- **10 pts** No Submission
- **8 pts** Failure to set up integral beyond the definition of moment of inertia

QUESTION 7

7 Problem 7 25 / 25

✓ **- 0 pts Everything is correct.**

- **1 pts** Parameterizations of x and y switched in part 1.
- **2 pts** Incorrect parameterization of x.
- **3 pts** Incorrect parameterization.
- **2 pts** Incorrect T_theta.
- **4 pts** Incorrect T_theta and T_phi.
- **4 pts** Solution to the second part of the problem missing.
- **1 pts** Incorrect sign of the third component of N.
- **1 pts** Incorrect signs of the second and third components of N.
- **3 pts** Incorrect N.
- **4 pts** Very incorrect ||N||.
- **6 pts** Very incorrect computation of N.
- **6 pts** Solution to the third part of the problem missing.
- **2 pts** Incorrect ||N||.
- **5 pts** Solution to the fourth part of the problem missing.
- **2 pts** Integration of incorrect ||N|| accidentally results in the correct answer.
- **5 pts** Very incorrect final result.
- **5 pts** Solution to the last part of the problem missing.
- **25 pts** Problem not attempted.
- **0 pts** Solution to the last part of the problem is very incorrect
- **4 pts** Click here to replace this description.
- **5 pts** Click here to replace this description.

QUESTION 8

8 Problem 8 **15 / 15**

✓ **- 0 pts Everything is correct.**

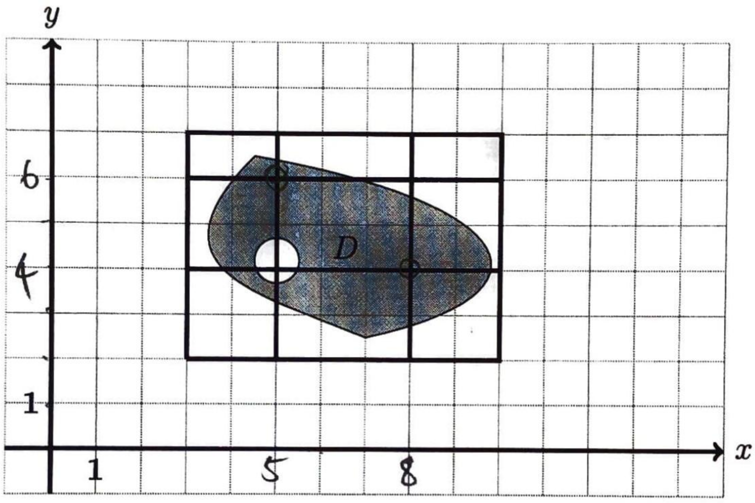
- **2 pts** Green's Theorem is incorrectly used in the first part.
- **5 pts** First part incorrect.
- **5 pts** First part not attempted.
- **4 pts** Gap in the second part argument.
- **8 pts** Your were supposed to isolate the singularity using part 1.
- **10 pts** Didn't use Green's Theorem.
- **10 pts** Second part incorrect.
- **10 pts** Second part not attempted.

Problem 1 **10 pts**

- Use the upper-right vertices of the below partition to find the Riemann sum $S_{3,3}$ for the integral

$$\iint_D (x^2 - y^2)\,dA$$

over the domain D shaded on the picture below. **8 pts**



$$S_{3,3} = f(5,6)\Delta A + f(8,4)\Delta A$$

$$= (5^2-6^2)\times(2\times2) + (8^2-4^2)\times(3\times2)$$

$$= -44 + 288$$

$$= 244$$

- What is the maximal length $||P||$ of the partition? **2 pts**

$||P|| =$ 3

1 Problem 1 **10 / 10**

✓ **- 0 pts** **Correct $$S_{3,3} = 244$$ with correct work**

✓ **- 0 pts** **Correct $$\|P\| = 3$$**

- 2 pts Areas incorrect

- 4 pts Did not multiply function values by areas

- 3 pts Did not take into account shape of integration domain by excluding test points that lie outside of it

- 2 pts Used evenly spaced points instead of given partition

- 2 pts Incorrect $$\|P\|$$

- 1 pts Included test point (5,4) in hole in domain

Problem 2 **10 pts**

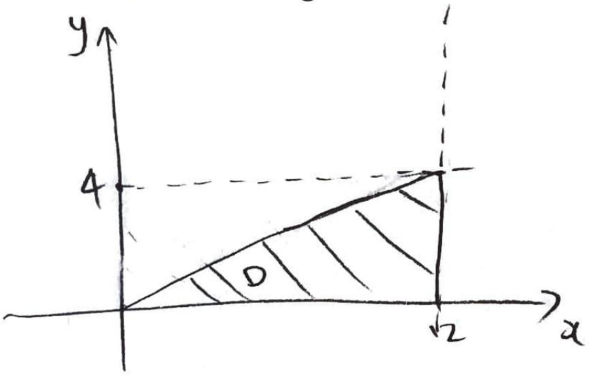
Consider the following integral.

$$\int_0^4 \int_{3y}^{12} e^{x^2}\, dx\, dy$$

$x=12$

$x=3y \Rightarrow y=\frac{x}{3}$

- Sketch the domain of integration. **2 pts**



- Switch the order of integration and evaluate. **8 pts**

$$\int_0^{12}\int_0^{x/3} e^{x^2}\, dy\, dx = \int_0^{12} \left[y e^{x^2} \right]_0^{\frac{x}{3}} dx$$

$$= \frac{1}{2\times 3}\int_0^{12} 2x e^{x^2}\, dx = \frac{1}{6}\left[e^{x^2} \right]_0^{12}$$

$$= \frac{1}{6}\left(e^{144} - 1 \right)$$

2 Problem 2 10 / 10

✓ **- 0 pts** **Accurate sketch**

✓ **- 0 pts** **Correct integral = $\frac{1}{6}(e^{144} - 1)$ with correct work**

- 1 pts Arithmetic errors

- 1 pts Incorrect bound on new integral

- 1 pts Triangle backwards in sketch

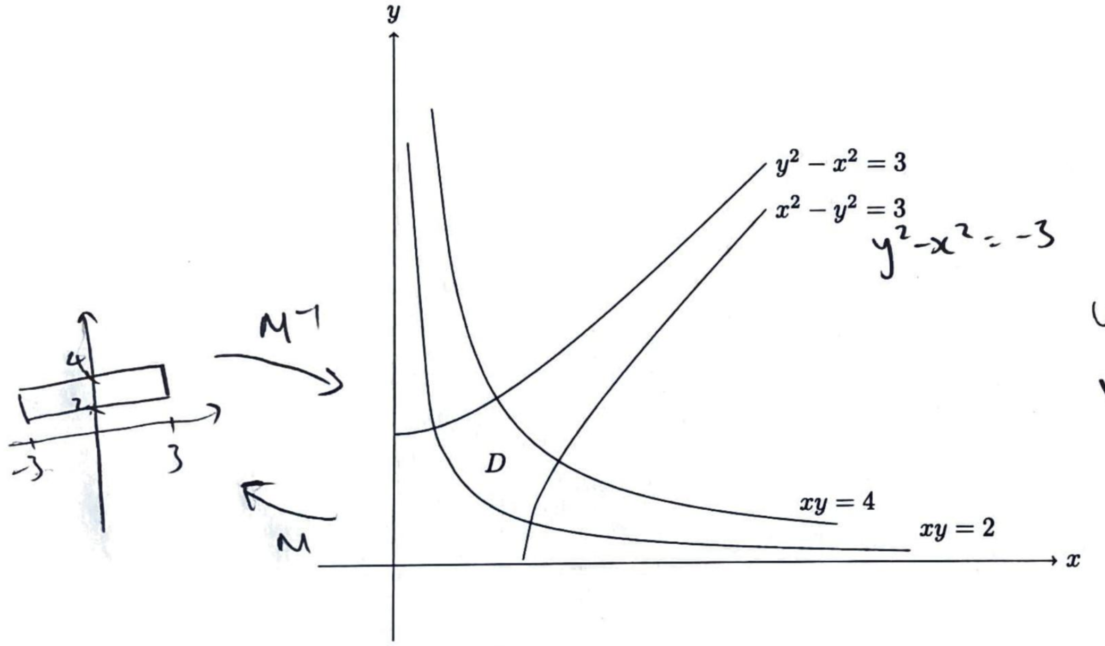
- 1 pts Sketched line $x=3y$ but not domain of integration

- 2 pts Incorrect sketch

Problem 3 **10 pts**

D is the region bounded by the curves on the picture below.
Find the following integral.

$$\iint\limits_D (x^2 + y^2)\,dA$$



$u = y^2 - x^2$

$v = xy$

$$Jac(M) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -2x & 2y \\ y & x \end{vmatrix} = -2x^2 - 2y^2$$

take abs value of

$Jac = (2x^2 + 2y^2)$

$Jac^{-1}(M) = \frac{1}{2x^2+2y^2}$

$$\Rightarrow \iint\limits_D x^2+y^2\,dA \Rightarrow \int_{-3}^{3}\int_{2}^{4} \frac{x^2+y^2}{2(x^2+y^2)}\,dv\,du$$

$$= \int_{-3}^{3}\int_{2}^{4} \frac{1}{2}\,dv\,du$$

$$= \frac{1}{2}\int_{-3}^{3} [v]_2^4\,du = \int_{-3}^{3} du = 6$$

3 Problem 3 **10 / 10**

- ✓ **- 0 pts** **Correct integral (6) with correct work**
- **- 1.5 pts** Off by minus sign: change of variables factor is absolute value of determinant
- **- 2.5 pts** Wrong bounds of new integral
- **- 0.5 pts** One wrong bound on new integral
- **- 0.5 pts** Arithmetic error
- **- 8 pts** Incorrect, but some work

$$\int_0^{10}\int_0^{10} 0.01\,dy\,dx = \int_0^{10}\left[0.01y\right]_0^{10}dx = \int_0^{10} 0.1\,dx$$
$$= 1$$

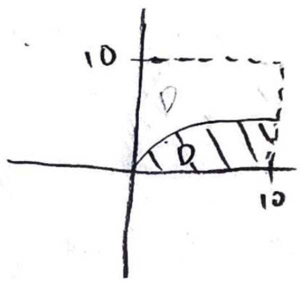
Problem 4 **10 pts**

The real numbers X and Y are randomly and independently chosen between zero and ten. The joint probability density is

$$p(x,y) = \begin{cases} 0.01 & \text{if } (x,y) \in [0,10]\times[0,10] \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability P that $X \geq Y^2$.

$x = y^2$



$$\int_0^{10}\int_0^{\sqrt{x}} 0.01\,dy\,dx$$

$$= \int_0^{10}\left[0.01\right]_0^{\sqrt{x}} dx = \int_0^{10} 0.01\sqrt{x}\,dx$$

$$= \left[\tfrac{2}{3}(0.01)x^{\frac{3}{2}}\right]_0^{10} = \tfrac{2}{3}(0.01)(10)^{\frac{3}{2}} = 0.211 \text{ (3sf)}$$

4 Problem 4 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 2 pts** Misinterpreting the $$X\geq Y^2$$ condition
- **- 2 pts** Giving an answer that could not be a probability (A negative number or a number more than 1)
- **- 1 pts** Arithmetic Mistake
- **- 10 pts** No Answer Given

Problem 5 **10 pts**

Find the volume of C. The answer without the derivation yields only one point!

From diagram on right.

$$\mathrm{Vol}(C) = \int_0^{2\pi}\int_0^h\int_0^{b-\frac{(b-a)z}{h}} 1\, r\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^h \left[\frac{r^2}{2}\right]_0^{b-\frac{(b-a)z}{h}} dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^h \frac{\left(b-\frac{(b-a)z}{h}\right)^2}{2}\, dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^h b^2 - \frac{2b(b-a)z}{h} + \frac{(b-a)^2z^2}{h^2}\, dz\,d\theta$$

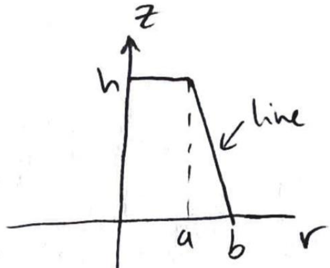
$$= \frac{1}{2}\int_0^{2\pi}\left[b^2z - \frac{b(b-a)z^2}{h} + \frac{(b-a)^2z^3}{3h^2}\right]_0^h d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta\left[b^2h - b(b-a)h + \frac{(b-a)^2h}{3}\right]$$

$$= \pi\left[b^2h - b^2h + abh + \frac{b^2h-2abh+a^2h}{3}\right]$$

$$= \frac{\pi}{3}\left[b^2h+abh+a^2h\right]$$

$$= \frac{\pi h}{3}\left[b^2+ab+a^2\right]$$



$y = x\cdot m + b$

line: $0 = b\left(\frac{h}{a-b}\right) + c$

$c = \frac{-bh}{a-b}$

line: $z = \left(\frac{h}{a-b}\right)r + \frac{bh}{b-a}$

$= -\left(\frac{h}{b-a}\right)r + \frac{bh}{b-a}$

$\left(\frac{h}{b-a}\right)r = \frac{bh}{b-a} - z$

$r = \frac{\frac{bh}{b-a}}{\frac{h}{b-a}} - \frac{z}{\frac{h}{b-a}}$

$r = b - \frac{(b-a)z}{h}$

5 Problem 5 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 6 pts** Major mistakes in setting up the integral
- **- 3 pts** Major mistakes in evaluating the integral
- **- 6 pts** Major parts of the problem left unsolved

Problem 6 **10 pts**

Recall that the cone is homogeneous and has mass M. Find the cone's moment of inertia with respect to the z-axis, I_z.

$$I_z = \iiint_W (x^2+y^2)\,\mu\, dV; \quad \mu = \frac{M}{\text{Vol}} = \frac{M}{\frac{\pi h}{3}[b^2+ab+a^2]}$$

$$I_z = \frac{3M}{\pi h[b^2+ab+a^2]} \int_0^{2\pi}\int_0^h\int_0^{b-\frac{(b-a)z}{h}} r^2 \; r\,dr\,dz\,d\theta$$

$$= \frac{3M}{\pi h[b^2+ab+a^2]} \int_0^{2\pi} d\theta \int_0^h \left[\frac{r^4}{4}\right]_0^{b-\frac{(b-a)z}{h}} dz$$

$$= \frac{6M}{h[b^2+ab+a^2]} \times \frac{1}{4} \int_0^h \left(b - \frac{(b-a)z}{h}\right)^4 dz = \frac{3M}{2h[b^2+ab+a^2]} \left[\frac{1}{5} \times \frac{-h}{b-a}\left(b - \frac{(b-a)z}{h}\right)^5\right]_0^h$$

$$= \frac{3M}{10[b^2+ab+a^2]} \left[\frac{-1}{b-a}\left[\left(b - \frac{(b-a)h}{h}\right)^5 - \left(b - \frac{(b-a)0}{h}\right)^5\right]\right]$$

$$= \frac{3M}{10[b^2+ab+a^2]} \cdot \frac{-1}{b-a}\left[(b-(b-a))^5 - (b-0)^5\right]$$

$$= \frac{3M}{10[b^2+ab+a^2]} \cdot \frac{-1}{b-a}\left[a^5 - b^5\right] = \frac{3M}{10[b^2+ab+a^2](b-a)}(b^5-a^5)$$

$$= \frac{3M(b^5-a^5)}{10[b^3+\cancel{ab^2}+\cancel{a^2b}-\cancel{ab^2}-\cancel{a^2b}-a^3]} = \frac{3M}{10} \times \frac{b^5-a^5}{b^3-a^3}$$

8

6 Problem 6 **10 / 10**

✓ **- 0 pts** Correct

- 5 pts Incorrectly setting up the integral, or setting up an integral for the wrong moment of Inertia

- 3 pts Incorrectly reading the instructions

- 1 pts Arithmetic Mistake

- 3 pts Failing to incorporate the density in the final answer

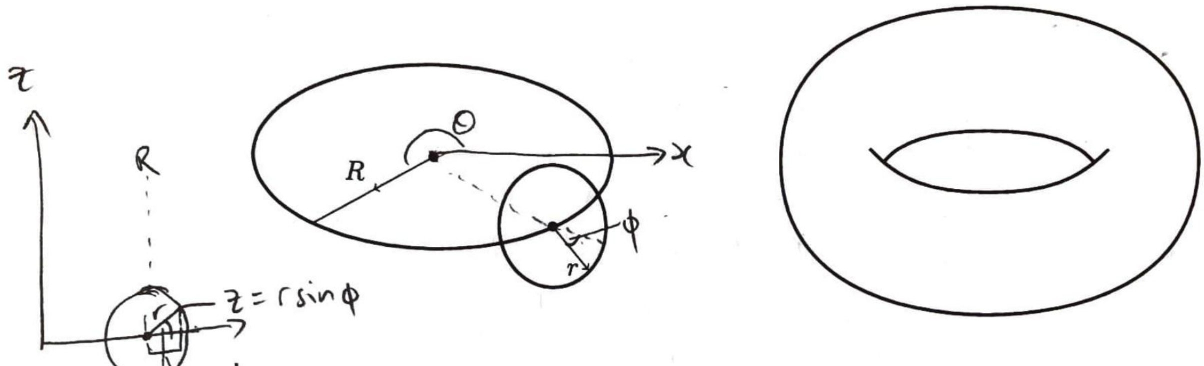
- 3 pts Mistakes in the Evaluation of the integral

- 10 pts No Submission

- 8 pts Failure to set up integral beyond the definition of moment of inertia

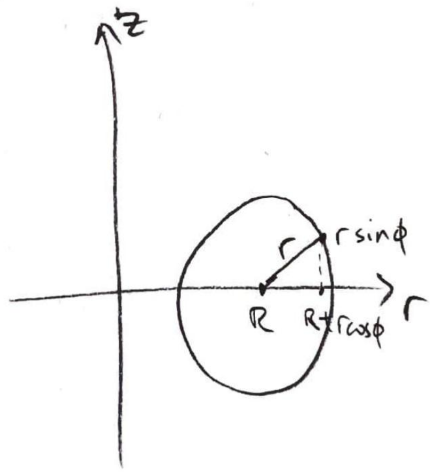
Problem 7 **25 pts**

A torus $T(R, r)$ is a 2D surface spanned by a circumference of radius r, its center rotating around a circumference of radius $R > r$, the plane of the smaller circumference being always perpendicular to the larger circumference.



- Assuming that the larger circumference lies in the xy-plane and is centered at the origin, find the parametrization $(x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi))$ of the torus where $0 \leq \theta < 2\pi$ is the angle the radius-vector of the larger circumference forms with the x-axis and $0 \leq \varphi < 2\pi$ is the angle the radius-vector of the smaller circumference forms with the xy-plane as shown on the left-hand side picture above. **5 pts**

$$z = r\sin\phi$$
$$r = R + r\cos\phi$$
$$\Rightarrow x = (R + r\cos\phi)\cos\theta$$
$$y = (R + r\cos\phi)\sin\theta$$
$$z = r\sin\phi$$



The problem continues to the next page.

- Find the vectors $\vec{T_\theta}$ and $\vec{T_\varphi}$. **4 pts**

$$T_\theta = \begin{bmatrix} -(R+r\cos\phi)\sin\theta \\ (R+r\cos\phi)\cos\theta \\ 0 \end{bmatrix}$$

$$T_\phi = \begin{bmatrix} -r\sin\phi\cos\theta \\ -r\sin\phi\sin\theta \\ r\cos\phi \end{bmatrix}$$

- Find the normal vector $\vec{N} = \vec{T_\theta} \times \vec{T_\varphi}$. **6 pts**

$$\vec{N} = \begin{vmatrix} i & j & k \\ -(R+r\cos\phi)\sin\theta & (R+r\cos\phi)\cos\theta & 0 \\ -r\sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi \end{vmatrix}$$

$$= \left[(R+r\cos\phi)\cos\theta\, r\cos\phi\right] i + \left[(R+r\cos\phi)\sin\theta\, r\cos\phi\right] j + \left[(R+r\cos\phi) r\sin\phi\sin^2\theta + (R+r\cos\phi) r\sin\phi\cos^2\theta\right] k$$

$$\vec{N} = \begin{bmatrix} (R+r\cos\phi)\cos\theta\, r\cos\phi \\ (R+r\cos\phi)\sin\theta\, r\cos\phi \\ (R+r\cos\phi) r\sin\phi \end{bmatrix}$$

The problem continues to the next page.

- Find the length of $\vec{N}$. **5 pts**

$$\|\vec{N}\| = \sqrt{(R+r\cos\phi)^2 r^2\cos^2\phi\cos^2\theta + (R+r\cos\phi)^2 r^2\cos^2\phi\sin^2\theta + (R+r\cos\phi)^2 r^2\sin^2\phi}$$

$$= \sqrt{(R+r\cos\phi)^2 r^2\cos^2\phi + (R+r\cos\phi)^2 r^2\sin^2\phi}$$

$$= \sqrt{r^2(R+r\cos\phi)^2}$$

$$= r(R+r\cos\phi)$$

The problem continues to the next page.

- Find the area of the torus. The correct answer unsupported by work yields one point only. **5 pts**

$$\text{area} = \int_0^{2\pi}\int_0^{2\pi} r(R + r\cos\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2\pi} rR + r^2\cos\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \Big[rR\phi + r^2\sin\phi \Big]_0^{2\pi} d\theta$$

$$= \int_0^{2\pi} rR(2\pi) + r^2(0) - 0\, d\theta$$

$$= 2\pi rR \int_0^{2\pi} d\theta$$

$$= 4\pi^2 rR$$

7 Problem 7 **25 / 25**

- ✓ **- 0 pts** **Everything is correct.**
- **- 1 pts** Parameterizations of x and y switched in part 1.
- **- 2 pts** Incorrect parameterization of x.
- **- 3 pts** Incorrect parameterization.
- **- 2 pts** Incorrect T_theta.
- **- 4 pts** Incorrect T_theta and T_phi.
- **- 4 pts** Solution to the second part of the problem missing.
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- **- 2 pts** Integration of incorrect ||N|| accidentally results in the correct answer.
- **- 5 pts** Very incorrect final result.
- **- 5 pts** Solution to the last part of the problem missing.
- **- 25 pts** Problem not attempted.
- **- 0 pts** Solution to the last part of the problem is very incorrect
- **- 4 pts** Click here to replace this description.
- **- 5 pts** Click here to replace this description.

Problem 8 **15 pts**

- Find the circulation of the vortex field

$$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

along a positively oriented circumference C_1 of radius r centered at the origin. **5 pts**

$\oint_C \vec{F} \cdot \vec{p}\, dt$, $r(t) = \langle r\cos t, r\sin t \rangle$

$r'(t) = \langle -r\sin t, r\cos t \rangle$

$F(r(t)) = \left\langle -\frac{r\sin t}{r^2}, \frac{r\cos t}{r^2} \right\rangle$

$$\oint_C \vec{F} \cdot \vec{p}\, dt = \int_0^{2\pi} \left\langle -\frac{r\sin t}{r^2}, \frac{r\cos t}{r^2} \right\rangle \cdot \langle -r\sin t, r\cos t \rangle\, dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt$$

$$= \int_0^{2\pi} 1\, dt$$

$$= 2\pi$$

The problem continues to the next page.

Assuming F is vortex field.

- Let C_2 be a smooth, simple, positively oriented path around the origin in the (x, y)-plane as on the picture below. Use Green's Theorem to find the following integral. **10 pts**

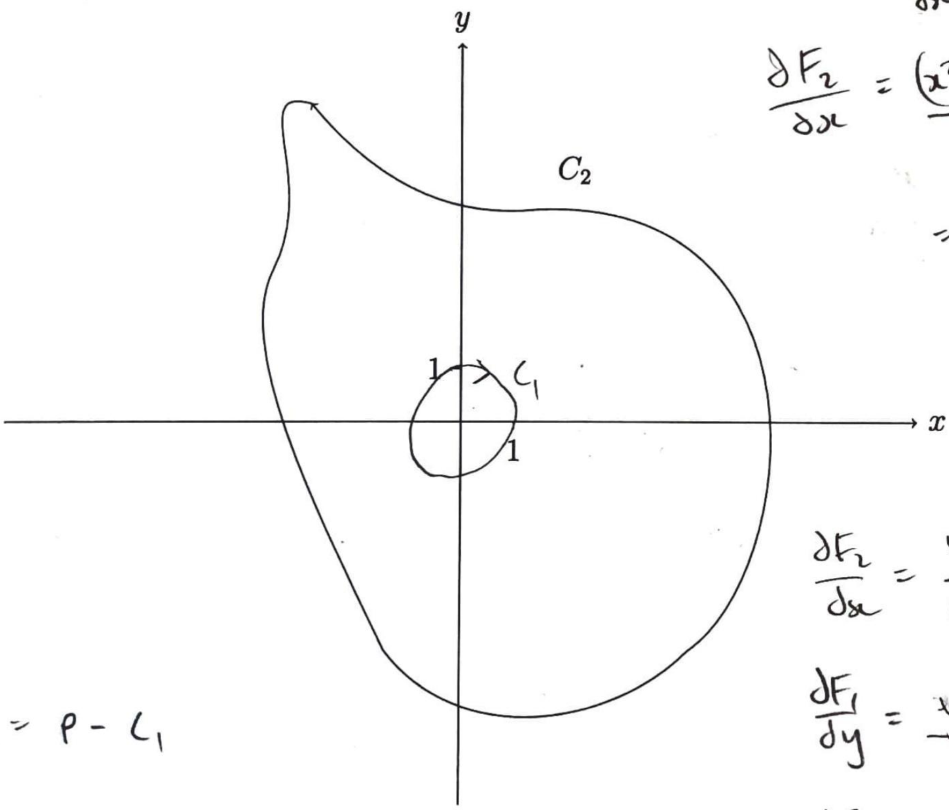
$$\oint_{C_2} \vec{F} \cdot d\vec{p}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{y^2+x^2}\right)$$

$$\frac{\partial F_2}{\partial x} = \frac{(x^2+y^2) - x\frac{d}{dx}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$



$$\frac{\partial F_2}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\partial C_2 = P - C_1$

$$\therefore \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

$$\oint_{\partial D} \vec{F} \cdot d\vec{p} = \oint_{C_2 - C_1} \vec{F}\, d\vec{p}.$$

$$= \oint_{C_2} \vec{F} \cdot d\vec{p} - \oint_{C_1} \vec{F} \cdot d\vec{p} = \oint_{C_2} \vec{F} \cdot d\vec{p} - 2\pi \text{ (from part 1)}$$

$$= \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = 0$$

$$\Rightarrow \oint_{C_2} \vec{F} \cdot d\vec{p} - 2\pi = 0$$

$$\therefore \oint_{C_2} \vec{F} \cdot d\vec{p} = 2\pi$$

8 Problem 8 **15 / 15**

- ✓ **- 0 pts** **Everything is correct.**
- **- 2 pts** Green's Theorem is incorrectly used in the first part.
- **- 5 pts** First part incorrect.
- **- 5 pts** First part not attempted.
- **- 4 pts** Gap in the second part argument.
- **- 8 pts** Your were supposed to isolate the singularity using part 1.
- **- 10 pts** Didn't use Green's Theorem.
- **- 10 pts** Second part incorrect.
- **- 10 pts** Second part not attempted.