Course 32 B Sec. 2 UCLA Department of Mathematics

Fall 2020 Instructor: Oleg Gleizer

# Final Exam

Please print your name and student ID in the designated space below. Show your work! Answers unsupported by work yield no credit.

Student's Name, First:____________________ Last:______________________

Student ID:______________________________

**Pledge:** I assert, on my honor, that I have not received assistance of any kind from any other person while working on the final and that I have not used any non-permitted materials or technologies during the period of this evaluation.

**Student's signature:** ______________________________________________

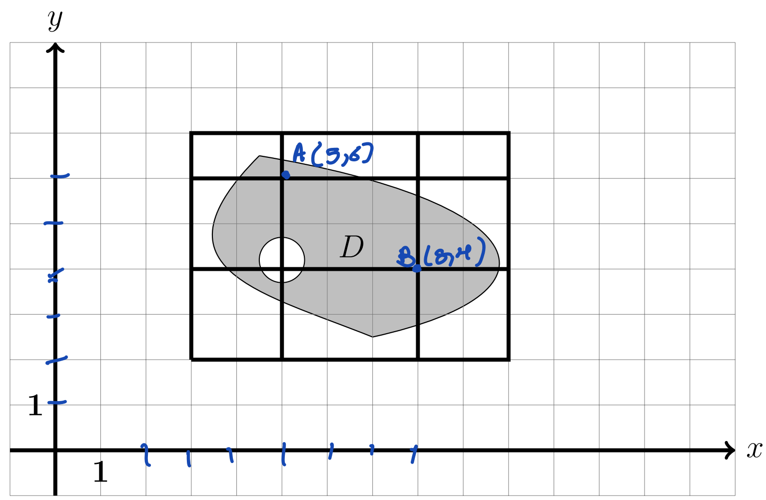
| Pr 1 | Pr 2 | Pr 3 | Pr 4 | Pr 5 | Pr 6 | Pr 7 | Pr 8 | Total |
|---|---|---|---|---|---|---|---|---|
| $\overline{10}$ | $\overline{10}$ | $\overline{10}$ | $\overline{10}$ | $\overline{10}$ | $\overline{10}$ | $\overline{25}$ | $\overline{15}$ | $\overline{100}$ |

**Problem 1** **10 pts**

- Use the upper-right vertices of the below partition to find the Riemann sum $S_{3,3}$ for the integral

$$\iint_D (x^2 - y^2)dA$$

over the domain $D$ shaded on the picture below. **8 pts**




We consider a point only if it lies in the domain. All other points have a value of 0.

$f(x,y) = x^2 - y^2$

$S_{3,3} =$ $f(A) \cdot \text{areaOfRectAt}(A) + f(B) \cdot \text{areaOfRectAt}(B)$

$= f(5,6) \cdot 4 + f(8,4) \cdot 6$

$= -11 \cdot 4 + 48 \cdot 6$

$= 288 - 44$

$= 244$

- What is the maximal length $||P||$ of the partition? **2 pts**

$||P|| =$ 3 [max dimension of a partition]



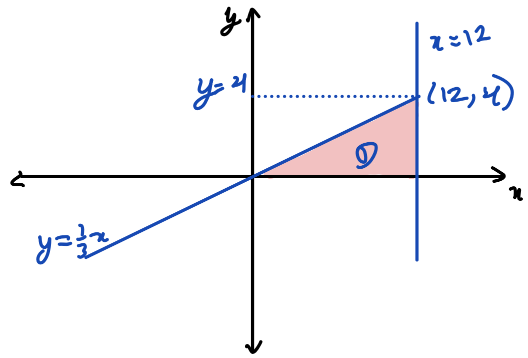
Since $||P|| = \max\limits_{\substack{1 \le i \le N \\ 1 \le j \le M}} (x_i - x_{i-1}, y_j - y_{j-1})$

**Problem 2** **10 pts**

Consider the following integral.

$$\int_0^4 \int_{3y}^{12} e^{x^2}\, dx\, dy$$

- Sketch the domain of integration. **2 pts**




D = domain of integration

- Switch the order of integration and evaluate. **8 pts**

Switching the order, we see $0 \leq y \leq \frac{1}{3}x$. $0 \leq x \leq 12$

$$\int_{x=0}^{12}\int_{y=0}^{\frac{1}{3}x} e^{x^2} dx = \int_{x=0}^{12} y e^{x^2}\Big|_{y=0}^{\frac{1}{3}x} dx = \frac{1}{3}\int_{x=0}^{12} x e^{x^2} dx$$

Let $u = x^2$, $du = 2x\,dx$

$$\frac{1}{6}\int_{u=0}^{144} e^u dx = \frac{1}{6}\left(e^u\right)_0^{144} = \frac{1}{6}\left(e^{144}-1\right)$$

**Problem 3** **10 pts**

$D$ is the region bounded by the curves on the picture below.
Find the following integral.

$$\iint_D (x^2 + y^2)\,dA$$

Let $u = y^2 - x^2$, $v = xy$.

$F(x,y) = (u(x,y), v(x,y))$

$= (y^2 - x^2, xy)$

Then

$\mathrm{Jac}\,F = \begin{vmatrix} -2x & y \\ 2y & x \end{vmatrix}$

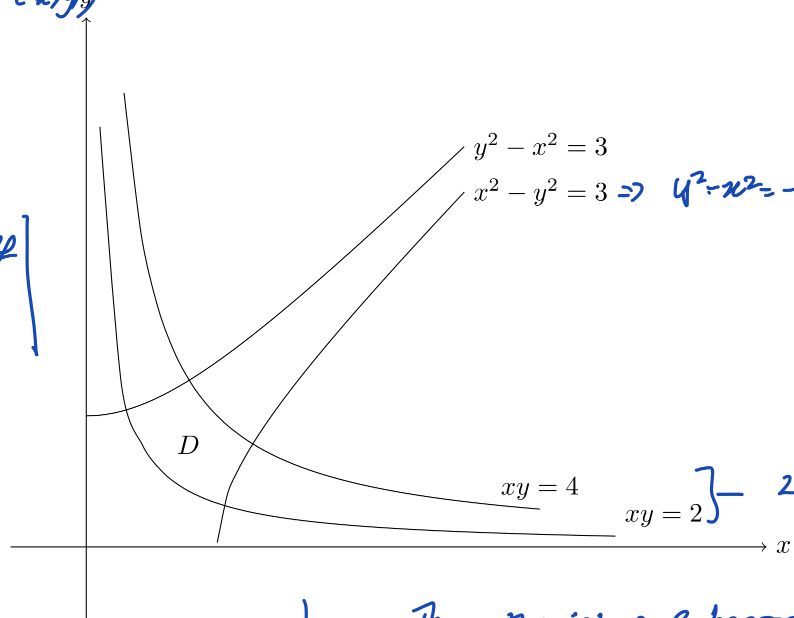
$= |-2x^2 - 2y^2|$

Since $x, y > 0$

$= 2x^2 + 2y^2$

$\mathrm{Jac}\,G = (\mathrm{Jac}\,F)^{-1}$ $[x, y \neq 0]$

$= \dfrac{1}{2(x^2+y^2)}$.




$x^2 - y^2 = 3 \Rightarrow y^2 - x^2 = -3$ $\Big\}$ $-3 \leq y^2 - x^2 \leq 3$

$xy = 2$ $\Big\}$ $2 \leq xy \leq 4$.

Then the integral becomes

$\iint_D (x^2+y^2) \cdot \dfrac{1}{2(x^2+y^2)}\,du\,dv$

$\frac{1}{2}\iint_D du\,dv = \frac{1}{2}\int_{u=2}^{4}\int_{v=-3}^{3} dv\,du$

$= \frac{1}{2}(4-2)(3+3)$

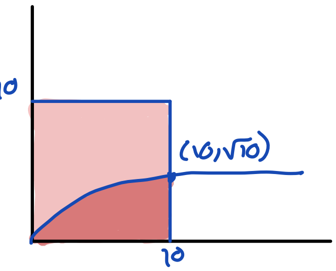
$= 6$

**Problem 4** **10 pts**

The real numbers $X$ and $Y$ are randomly and independently chosen between zero and ten. The joint probability density is

$$p(x,y) = \begin{cases} 0.01 & \text{if } (x,y) \in [0,10] \times [0,10] \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability $P$ that $X \geq Y^2$. $\Leftrightarrow \sqrt{x} \geq y$ as $x \geq 0$




We have the domain as shown.

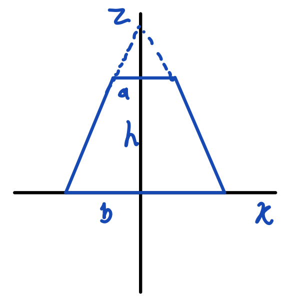
Then using the figure, we can calculate the domain as

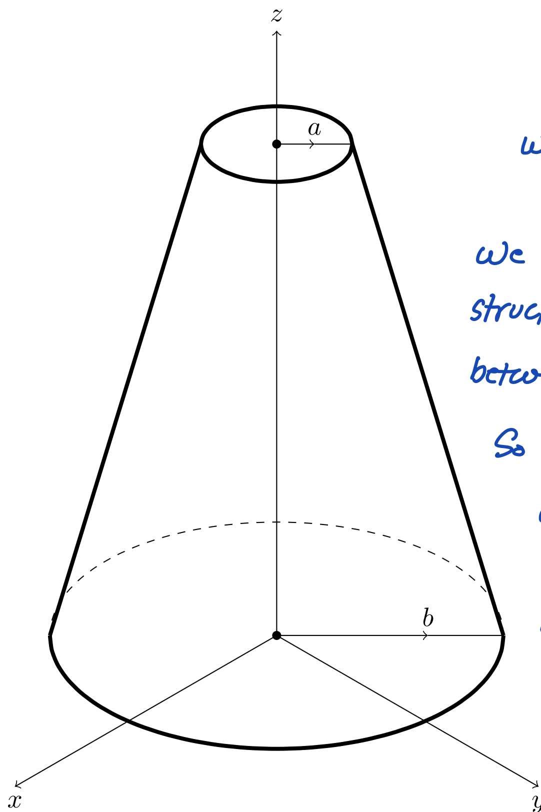
$$\int_{x=0}^{10}\int_{y=0}^{\sqrt{x}} p(x,y)\,dy\,dx = \int_{x=0}^{10}\int_{y=0}^{\sqrt{x}} 0.01\,dy\,dx = \int_{x=0}^{10} 0.01\,y\Big|_{y=0}^{\sqrt{x}}\,dx = \int_{x=0}^{10} 0.01\sqrt{x}$$

$$= \frac{2 \cdot 0.01\ x^{3/2}}{3}\Big|_{x=0}^{10} = \frac{0.02 \cdot x^{3/2}}{3}\Big|_{x=0}^{10} = \frac{0.02 \cdot 10^{3/2}}{3}$$

In Problems 5 and 6 of this test, you are asked to consider a truncated straight circular cone $\mathcal{C}$. Its lower base is a circle of radius $b$ located in the $xy$-plane and centered at the origin. Its upper base is a circle of radius $a$ located in the plane $z = h$ and centered at the point $P = (0, 0, h)$ as shown on the picture below. The cone is homogeneous of mass $M$.

In two dimensions, say the $x$-$z$ plane we have




when $r=b$, $z=0$

$r=a$, $z=h$.

We also know that for a conical structure the is a lineal relationship between $r$ and $z$.

So $r \propto z$

when $z=0$, $r=b$

$r = mz+b$

when $z=h$, $r=a$

$a = mh+b$

$m = \frac{a-b}{h}$

So $r = -\frac{b-a}{h}z+b$

**Problem 5** **10 pts**

Find the volume of $\mathcal{C}$. The answer without the derivation yields only one point!

As we proved above, $r : 0 \to -\frac{b-a}{h}z + b$, $z : 0 \to h$, $\theta : 0 \to 2\pi$

$$\int_{\theta=0}^{2\pi}\int_{z=0}^{h}\int_{r=0}^{-\frac{b-a}{h}z+b} r\,dr\,dz\,d\theta = \int_{z=0}^{h}\int_{r=0}^{-\frac{b-a}{h}z+b}\int_{\theta=0}^{2\pi} r\,d\theta\,dr\,dz$$

$$= \int_{z=0}^{h}\int_{r=0}^{-\frac{b-a}{h}z+b} 2\pi r\,dr\,dz = \pi\int_{z=0}^{h} r^2\Big|_0^{-\frac{b-a}{h}z+b} dz = \pi\int_{z=0}^{h}\left(b-\frac{b-a}{h}z\right)^2 dz$$

$$= \pi\int_{z=0}^{h} b^2 + \left(\frac{b-a}{h}\right)^2 z^2 - \frac{2b(b-a)}{h}z\,dz$$

$$= \pi\left(b^2 z + \frac{1}{3}\left(\frac{b-a}{h}\right)^2 z^3 - \frac{b(b-a)z^2}{h}\right)\Big|_0^h$$

$$= \pi\left(b^2h + \frac{1}{3}(b-a)^2h - b(b-a)h\right)$$

$$= \frac{\pi}{3}\left(3b^2h + (b-a)^2h - 3(b^2-ba)h\right)$$

$$= \frac{\pi h}{3}\left(3b^2 + b^2 + a^2 - 2ab - 3b^2 + 3ba\right)$$

$$= \frac{\pi h}{3}\left(b^2 + ab + a^2\right)$$

**Problem 6** **10 pts**

Recall that the cone is homogeneous and has mass $M$. Find the cone's moment of inertia with respect to the $z$-axis, $I_z$.

Let the density be $\rho = \frac{m}{V} =$ constant $= c$

We want to find $I_z = \iiint_\omega (x^2+y^2) c \, dV$

In cylindrical coordinates

$$I_z = c \int_{z=0}^{h} \int_{r=0}^{-\frac{(b-a)}{h}z+b} \int_{\theta=0}^{2\pi} r^2 \cdot r \, d\theta \, dr \, dz = 2\pi c \int_{z=0}^{h} \int_{r=0}^{-\frac{(b-a)}{h}z+b} r^3 dr \, dz$$

$$= \frac{\pi c}{2} \int_{z=0}^{h} r^4 \Big|_0^{-\frac{(b-a)}{h}z+b} dz$$

$$= \frac{\pi c}{2} \int_{z=0}^{h} \left( b - \frac{(b-a)z}{h} \right)^4 dz = \frac{\pi c}{2h^4} \int_{z=0}^{h} (bh + (a-b)z)^4 dz$$

Let $u = bh + (a-b)z$, then $du = (a-b)dz$

$$= \frac{\pi c}{2h^4(a-b)} \int_{u=bh}^{ah} u^4 dz$$

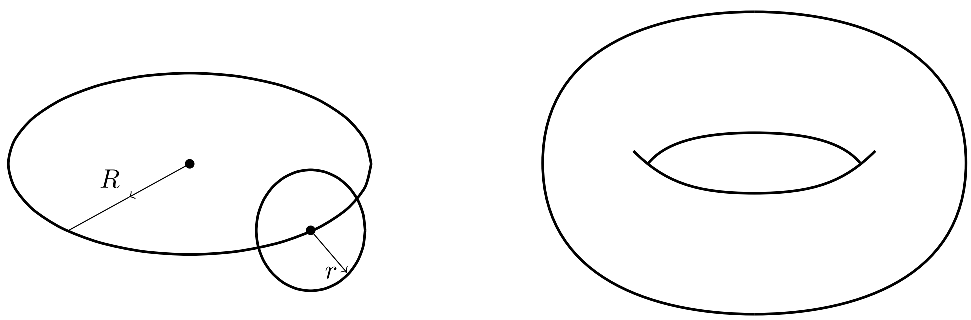
$$= \frac{\pi c}{2h^4(a-b)} \cdot \frac{u^5}{5} \Big|_{u=bh}^{ah} = \frac{\pi c}{10h^4(a-b)} \cdot (a^5h^5 - b^5h^5)$$

$$= \frac{\pi c h}{10(a-b)} (a^5-b^5) = \frac{\pi h (b^5-a^5)}{10(b-a)} \cdot \frac{3m}{\pi h (b^2+ab+a^2)} = \frac{3m(b^5-a^5)}{10(b-a)(b^2+ab+a^2)}$$

$$= \frac{3m(b^5-a^5)}{10(b^3-a^3)}$$

**Problem 7** **25 pts**

A torus $T(R, r)$ is a 2D surface spanned by a circumference of radius $r$, its center rotating around a circumference of radius $R > r$, the plane of the smaller circumference being always perpendicular to the larger circumference.




- Assuming that the larger circumference lies in the $xy$-plane and is centered at the origin, find the parametrization $(x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi))$ of the torus where $0 \leq \theta < 2\pi$ is the angle the radius-vector of the larger circumference forms with the $x$-axis and $0 \leq \varphi < 2\pi$ is the angle the radius-vector of the smaller circumference forms with the $xy$-plane as shown on the left-hand side picture above. **5 pts**

Along the xy plane we are forming a circle of radius R.

So we can initially say $x = R\cos\theta$, $y = R\sin\theta$

At every point in the x,y plane, we are doing a rotation in the plane along the radius vector for the larger circle.

We know for a fact that since rotation in xy plane has no effect of z,

$z = r\sin\varphi$

Now in x and y direction, there will be a term of $r\cos\varphi$.

The problem continues to the next page.

It would depend on the position of $\theta$. Since, if $\theta = 0$, rotation is restricted in the x-z axis and for $\theta = \frac{\pi}{2}$, rotation is restricted in z-y axis. The center of this circle is also dependant on the position of $\theta$.

So. $x = R\cos\theta + \cos\theta \cdot r\cos\varphi = \cos\theta(R + r\cos\varphi)$

$y = R\sin\theta + \sin\theta\, r\sin\varphi = \sin\theta(R + r\cos\varphi)$

Thus we get the parameterization

$$G(\theta,\varphi) = (\cos\theta(R+r\cos\varphi), \sin\theta(R+r\cos\varphi), r\sin\varphi)$$

- Find the vectors $\vec{T}_\theta$ and $\vec{T}_\varphi$. **4 pts**

$$\vec{T}_\theta = \frac{\partial G}{\partial \theta} = \langle -\sin\theta(R+r\cos\varphi), \cos\theta(R+r\cos\varphi), 0\rangle$$

$$\vec{T}_\varphi = \frac{\partial G}{\partial \varphi} = \langle -r\sin\varphi\cos\theta, -r\sin\varphi\sin\theta, r\cos\varphi\rangle$$

- Find the normal vector $\vec{N} = \vec{T}_\theta \times \vec{T}_\varphi$. **6 pts**

$$\vec{N}(\theta,\varphi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta(R+r\cos\varphi) & \cos\theta(R+r\cos\varphi) & 0 \\ -r\sin\varphi\cos\theta & -r\sin\varphi\sin\theta & r\cos\varphi \end{vmatrix}$$

$$= \hat{i}(r\cos\varphi\cos\theta(R+r\cos\varphi)) - \hat{j}(-r\cos\varphi\sin\theta(R+r\cos\varphi)) + \hat{k}(r\sin^2\theta\sin\varphi(R+r\cos\varphi) + r\sin\varphi\cos^2(R+r\cos\varphi))$$

$$= \langle r\cos\varphi\cos\theta(R+r\cos\varphi), r\cos\varphi\sin\theta(R+r\cos\varphi), r\sin\varphi(R+r\cos\varphi)\rangle$$

$$= r(R+r\cos\varphi)\langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi\rangle$$

The problem continues to the next page.

- Find the length of $\vec{N}$. **5 pts**

Length of $\vec{N} = \|\vec{N}\| = r(R+r\cos\varphi) \| \langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi \rangle \|$

$= r(R+r\cos\varphi)\sqrt{\cos^2\varphi\cos^2\theta + \cos^2\varphi\sin\theta + \sin^2\varphi}$

$= r(R+r\cos\varphi)\sqrt{\cos^2\varphi + \sin^2\varphi}$

$= r(R+r\cos\varphi)$

The problem continues to the next page.

- Find the area of the torus. The correct answer unsupported by work yields one point only. **5 pts**

Area of torus is given by

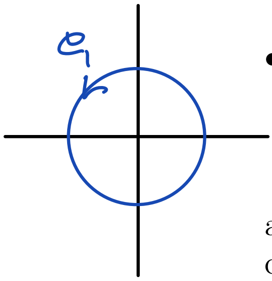
$$\iint_{\mathcal{S}} dS = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} \|\vec{N}\|\, d\varphi\, d\theta = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} r(R+\cos\varphi)\, d\varphi\, d\theta$$

$$= \int_{\theta=0}^{2\pi} Rr\varphi + r\sin\varphi \Big|_{\varphi=0}^{2\pi} d\theta$$

$$= \int_{\theta=0}^{2\pi} 2\pi Rr\, d\theta$$

$$= 4\pi^2 Rr$$

**Problem 8** **15 pts**




- Find the circulation of the vortex field

$$\vec{F} = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

along a positively oriented circumference $C_1$ of radius $r$ centered at the origin. **5 pts**

We want to find $\oint_{C_1} \vec{F} \cdot d\vec{r}$. However, we can not apply Green's theorem here because the point $(0,0)$ [which is undefined for the vortex field] is in the domain.

So instead, we will use polar coordinates. $\vec{r}(\theta) = \langle r\cos\theta, r\sin\theta \rangle$, $0 \le \theta \le 2\pi$

$\Rightarrow \vec{r}\,'(\theta) = \langle -r\sin\theta, r\cos\theta \rangle$

Then $\oint \vec{F} \cdot d\vec{r} = \oint \left\langle \frac{-r\sin\theta}{r^2}, \frac{r\cos\theta}{r^2} \right\rangle \cdot \langle -r\sin\theta, r\cos\theta \rangle \, d\theta$

$$= \int_{\theta=0}^{2\pi} \langle -\sin\theta, \cos\theta \rangle \cdot \langle \sin\theta, \cos\theta \rangle \, d\theta$$

$$= \int_{\theta=0}^{2\pi} (\sin^2\theta + \cos^2\theta) \, d\theta = 2\pi$$

The problem continues to the next page.

- Let $C_2$ be a smooth, simple, positively oriented path around the origin in the $(x, y)$-plane as on the picture below. Use Green's Theorem to find the following integral. **10 pts**

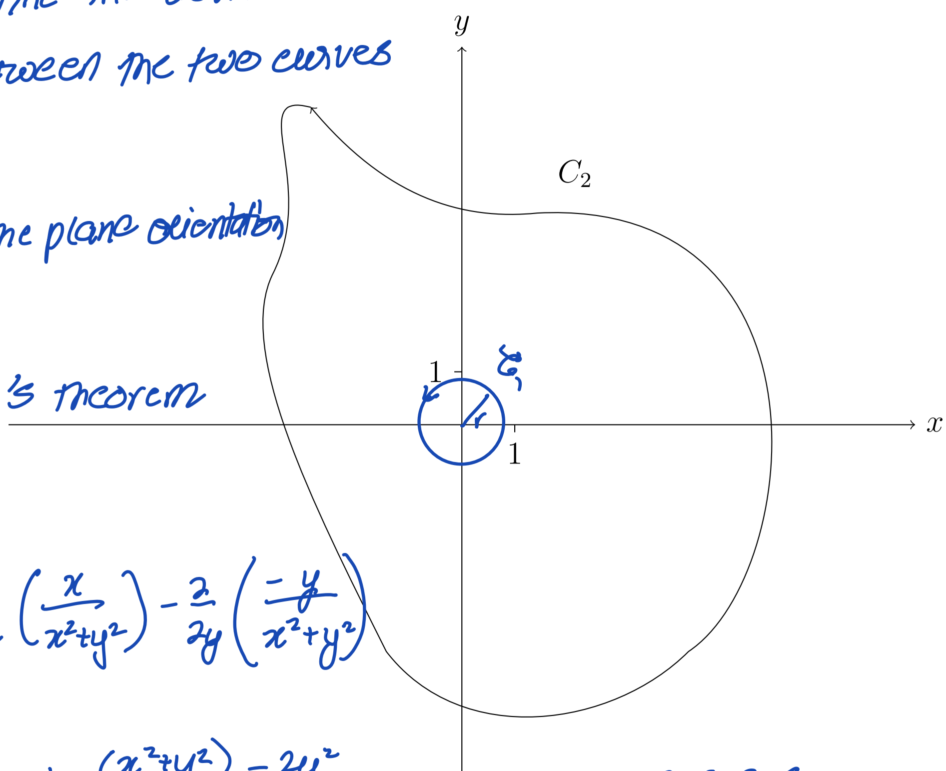
$$\oint_{C_2} \vec{F} \cdot d\vec{p}$$

Define $C_1$ to be a circle of radius $r$ with counterclock orientation

Then if we define the domain to be the area between the two curves

$\partial D = -C_1 \cup C_2$

(where outside the plane orientation is positive)




So using Greene's theorem

$\vec{F} = \langle F_1, F_2 \rangle$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} + \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{2x^2+2y^2-2x^2-2y^2}{(x^2+y^2)^2} = 0$$

Now $\oint_{\partial D} \vec{F} \cdot d\vec{p} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = 0$

$-\oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} = 0$

$\oint_{C_2} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r}$

Now, if we take $\lim_{r \to 0}$ for the circle $C_1$, then we will converge upon the value of $\oint_{C_2} \vec{F} \cdot d\vec{r}$ for the entire domain enclosed by $C_2$.

So, $\oint_{C_2} \vec{F} \cdot d\vec{r} = \lim_{r \to 0} \oint_{C_1} \vec{F} \cdot d\vec{r} = \lim_{r \to 0} 2\pi = 2\pi$

from first part