Midterm 2

UCLA: Math 32B, Winter 2017

Instructor: Noah White
Date: 27 February, 2017

- This exam has 5 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: [redacted]

ID number: [redacted]

Discussion section (please circle):

Day/TA	Ben	Gyu Eun	Robbie
Tuesday	3A	3C	3E
Thursday	3B	3D	3F

Question	Points	Score
1	9	8
2	8	6
3	7	2
4	5	1
5	11	7
Total:	40	24

+ 1

25

Here are some formulas that you may find useful as some point in the exam.

$$\int \cos^2 x \, dx = \frac{1}{2}(x + \cos x \sin x)$$

$$\int \sin^2 x \, dx = \frac{1}{2}(x - \cos x \sin x)$$

$$\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x$$

Spherical coordinates are given by

$$x(\rho, \theta, \phi) = \rho \cos\theta \sin\phi$$
$$y(\rho, \theta, \phi) = \rho \sin\theta \sin\phi$$
$$z(\rho, \theta, \phi) = \rho \cos\phi$$

The Jacobian for the change of coordinates is $J = \rho^2 \sin\phi$.

1. Let $\mathcal{E}$ be the solid region defined by

$$x^2 + y^2 + z^2 \leq a, \quad x, y, z \geq 0,$$

for a fixed constant $a > 0$. Suppose the region has a constant mass density of $\delta(x, y, z) = 1$.

(a) (2 points) Express the total mass of $\mathcal{E}$ as an iterated integral.

$\mathcal{E}$: sphere centered at origin with radius $= \sqrt{a}$

$\mathcal{E}$: $\langle \rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi \rangle$ $\quad \rho^2 = x^2+y^2+z^2$

$$\boxed{M = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta}$$

$0 \leq \rho \leq a$

$0 \leq \theta \leq \pi/2$

$0 \leq \phi \leq \pi/2$

$J = \rho^2 \sin\phi$

2 2

(b) (2 points) Find the total mass of $\mathcal{E}$.

$$M = \int_0^{\pi/2}\int_0^{\frac{\pi}{2}}\int_0^a \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta$$

$$M = \int_0^{\pi/2}\int_0^{\pi/2} \frac{a^3}{3} \sin\theta \, d\phi \, d\theta$$

$$M = \int_0^{\pi/2} \frac{\pi a^3}{6} \sin\theta \, d\theta$$

2

$$\cancel{M = 2\int_0^{\pi} \frac{\pi a^3}{3} \sin\theta \, d\theta}$$

$$M = \frac{\pi a^3}{6}\Big[-\cos\theta\Big]_0^{\pi/2}$$

$$\cancel{M = \frac{4\pi a^3}{3}}$$

$$\boxed{M = \frac{\pi a^3}{6}}$$

4

(c) (3 points) Express the coordinates of the center of mass of $\mathcal{E}$ as an iterated triple integral.

$x^2+y^2+z^2 \le a$ $\quad x,y,z \ge 0$

~~CM=~~

~~$0 \le x \le \sqrt{a-y^2-z^2}$~~

~~$0 \le y \le$~~

$0 \le x \le \sqrt{a}$

$0 \le y \le \sqrt{a-x^2}$

$0 \le z \le \sqrt{a-x^2-y^2}$

~~$CM: \int_0^{\sqrt{a-x^2-y^2}} \int_0^{\sqrt{a-x^2}} \int_0^{\sqrt{a}} (x,y,z)\, dx\, dy\, d$~~

$$\boxed{CM: \frac{6}{\pi a^3} \int_0^{\sqrt{a}} \int_0^{\sqrt{a-x^2}} \int_0^{\sqrt{a-x^2-y^2}} (x,y,z)\, dz\, dy\, dx}$$

3

(d) (2 points) Find the z coordinate of the center of mass.

$$M_z = \int_0^{\sqrt{a}} \int_0^{\sqrt{a-x^2}} \int_0^{\sqrt{a-x^2-y^2}} z \, dz\, dy\, dx$$

$$= \int_0^{\sqrt{a}} \int_0^{\sqrt{a-x^2}} \frac{a-x^2-y^2}{2}\, dy\, dx$$

$$= \int_0^{\sqrt{a}} \left(\frac{a-x^2}{2}\right)\sqrt{a-x^2} - \frac{y^3}{6}\Big|_0^{\sqrt{a-x^2}} dx$$

$$= \int_0^{\sqrt{a}} \frac{(a-x^2)^{3/2}}{2} - \frac{(a-x^2)^{3/2}}{6}\, dx$$

$$= \int_0^{\sqrt{a}} \frac{(a-x^2)^{3/2}}{3}\, dx$$

$$=$$

$$z = \frac{2}{\pi a^3} \int_0^{\sqrt{a}} (a-x^2)^{3/2}\, dx$$

alternative

$$\int_0^{\pi/2} \int_0^{\sqrt{a}} \frac{a-r^2}{2}\, dr\, d\theta$$

$$= \frac{\pi}{2}\left(\frac{a^{3/2}}{2} - \frac{a^{3/2}}{6}\right)$$

$$= \frac{2\pi a^{3/2}}{12}$$

$$\frac{4\sqrt{a}}{12}$$

$$z = \quad \boxed{= z = \frac{\sqrt{a}}{3}}$$

2. Consider the helix C, given by the parameterisation

$$\mathbf{r}(t) = \left(\cos t, \sin t, \frac{1}{2\pi} t\right) \quad t \in [0, 4\pi],$$

so that C is oriented with the z coordinate increasing.

(a) (4 points) Compute the length of C.

$$r'(t) = \left(-\sin t, \cos t, \frac{1}{2\pi}\right)$$

$$\| r'(t) \| = \sqrt{\sin^2 t + \cos^2 t + \frac{1}{4\pi^2}} = \sqrt{1 + \frac{1}{4\pi^2}}$$

$$\int_C 1 \, ds = \int_C \| r'(t) \| \, dt$$

$$= \int_0^{4\pi} \sqrt{1 + \frac{1}{4\pi^2}} \, dt$$

$$= 4\pi \sqrt{1 + \frac{1}{4\pi^2}}$$

$$\boxed{= 2\sqrt{4\pi^2 + 1}}$$

(b) (4 points) Compute the work done by the field

$$2zx + 2zy^2 - e^z$$

$$\mathbf{F}(x, y, z) = \langle z^2, 2yz^2, 2z(x + y^2) - e^z \rangle$$

on a particle constrained to move on the curve C.

$t \in [0, 4\pi]$

$r(t) = (\cos t, \sin t, \frac{1}{2\pi} t)$

$r'(t) = (-\sin t, \cos t, \frac{1}{2\pi})$

$$\int_C \underline{F} \cdot dr = \int_C F(r(t)) \cdot r'(t)\, dt$$

$\boxed{\text{conservative}}$

$$F(r(t)) = \left\langle \frac{t^2}{4\pi^2}, \frac{t^2 \sin t}{2\pi^2}, \frac{t}{\pi}(\cos t + \sin^2 t) - e^{t/2\pi} \right\rangle$$

$f = z^2 x + \alpha(y z)$

2

$f = y^2 z^2 + \beta(x z)$

$f = z^2 x + z^2 y^2 + e^z + \gamma(xy)$

$f = z^2 x + z^2 y^2 - e^z$ defined everywhere

$t \in [0, 4\pi]$

$f(t=0) = -1$

$r(0) = (1, 0, 0)$

$f(t=4\pi) = 4 - e^2$

$r(4\pi) = (1, 0, 2)$

$$\int_C \underline{F} \cdot dr = 5 - e^2$$

2/7

3. For this question consider the vector field

$$\mathbf{F}(x,y) = \frac{1}{r^2}\langle y(r^2-1), x(r^2+1)\rangle,$$

where $r = \sqrt{x^2+y^2}$. This vector field is defined everywhere apart from the origin.

(a) (4 points) Is $\mathbf{F}$ conservative on the domain described above? Justify your answer.

2 ~~8~~

F is not conservative on the above domain, because for a function to be conservative on a given domain ~~that~~ the curl must equal zero and the domain be simply connected of which the latter is not true or you must be able to find a potential function that is defined everywhere, but that can not happen since F is not defined everywhere.

$$F(x,y) = (y,x) + \left(\frac{-y}{r^2}, \frac{x}{r^2}\right)$$

$$= \nabla(xy) + \text{vortex}$$

0

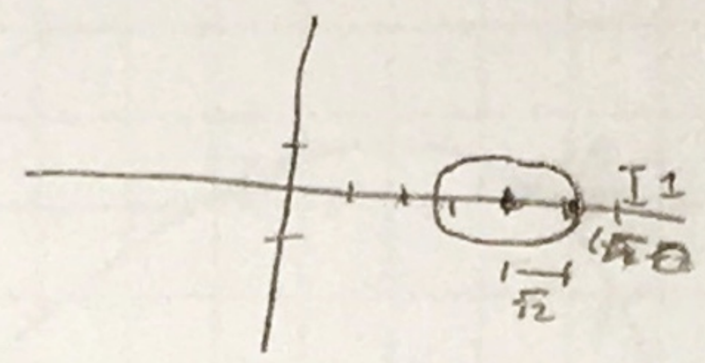
(b) (1 point) Give a domain on which $\mathbf{F}$ is conservative.

0

(c) (2 points) Calculate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{dr}$$

where C is the ellipse $\frac{(x-4)^2}{2} + y^2 = 1$, oriented in the counter clockwise direction.



$\boxed{2\pi}$ +

$\oint_C (y,x) \cdot dr + \int_C \text{vortex} \cdot dr$

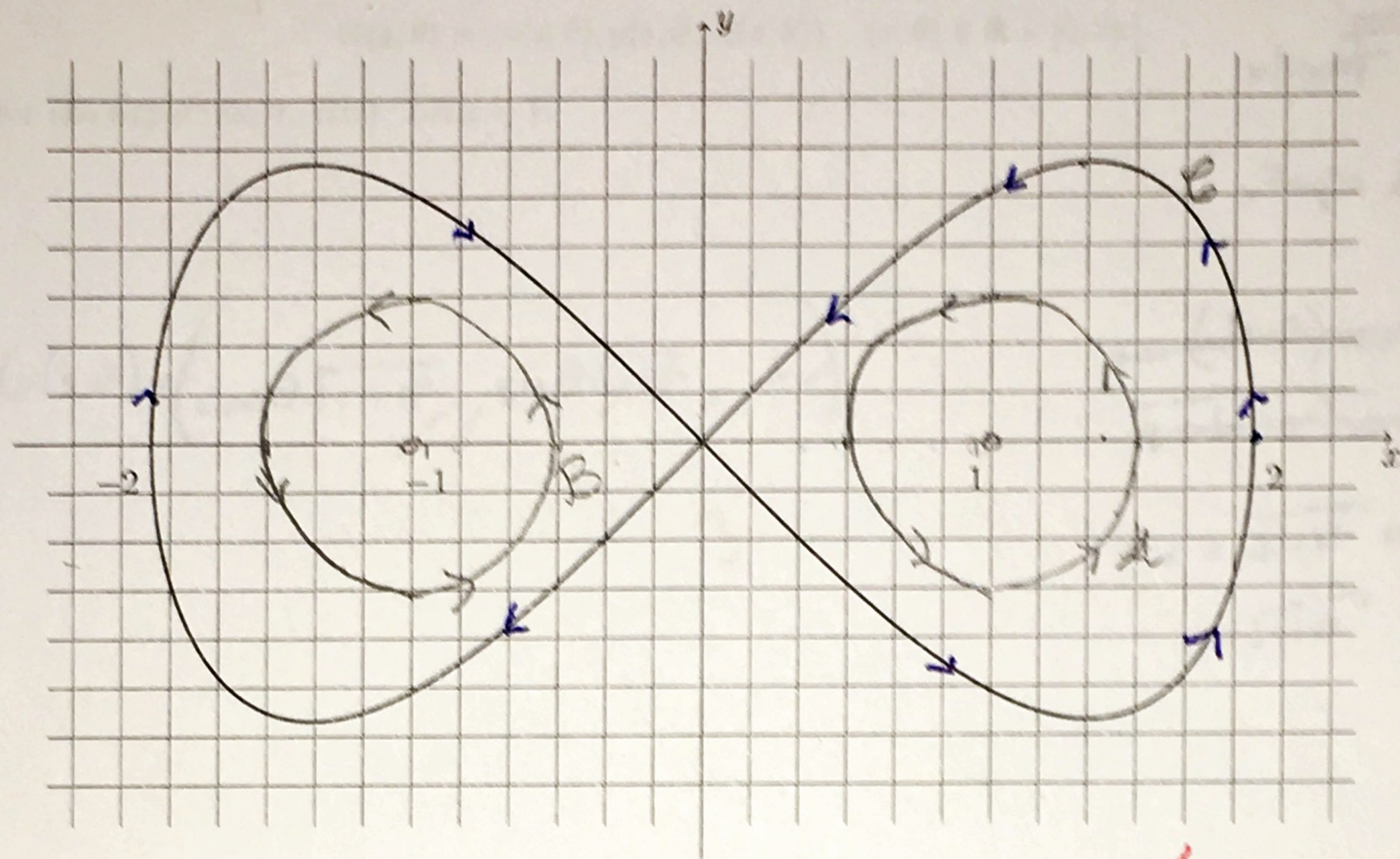
conservative

not around vortex

$= 0$

4. In this question assume that $\mathbf{E}$ is a vector field defined on the whole plane, apart from the points $(\pm 1, 0)$. The function $\mathbf{r}(t) = (2\cos t, \sin 2t)$ for $t \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ defines the curve C on the graph below

Curl(E) = 0



(a) (1 point) Indicate on the above graph, the orientation of the curve.

(b) (4 points) Let $\mathcal{A}$ and $\mathcal{B}$ be the circles, radius $\frac{1}{2}$, and center $(1, 0)$ and $(-1, 0)$ respectively, both oriented counter clockwise. Suppose that

$$\int_{\mathcal{A}} \mathbf{E} \cdot d\mathbf{r} = 2 \quad \text{and} \quad \int_{\mathcal{B}} \mathbf{E} \cdot d\mathbf{r} = 1.$$

What is $\int_C \mathbf{E} \cdot d\mathbf{r}$? Justify your answer.

5. The *hyperboloid* is Noah's favorite surface. It is given by the equation $x^2 + y^2 - z^2 = 1$.

(a) (3 points) Find a parameterisation

$$G(s, \theta) = (x(s, \theta), y(s, \theta), z(s, \theta)) \quad (s, \theta) \in \mathbb{R} \times [0, 2\pi]$$

for the hyperboloid. *Hint: Let $z = s$.*

x^2

$x^2+y^2 = 1+s^2$

$G(s,\theta) = (\cos\theta\sqrt{1+s^2}, \sin\theta\sqrt{1+s^2}, s)$

~~$x = (1+s^2)\cos\theta$~~

~~$y = (1+s^2)\sin\theta$~~

3

$x = \sqrt{1+s^2}\cos\theta$

$y = \sqrt{1+s^2}\sin\theta$

(b) (5 points) Express the surface area of the hyperboloid between $z = a$ and $z = -a$ as an iterated integral.

$-a \le z \le a$

$0 \le \theta \le 2\pi$ $\quad$ $-a \le s \le a$

$T_s = (\quad , 1)$

$T_\theta = (-\sin\theta\sqrt{1+s^2}, \cos\theta\sqrt{1+s^2}, 0)$

4

$SA = \int_0^{2\pi} \int_{-a}^{a} \|N(s,\theta)\| \, ds \, d\theta$

$(1+s^2)^{1/2}$

$s(1+s^2)^{-1/2}$

(extra working room for part (b))

(c) (3 points) Calculate the surface area.

0