Midterm 2

UCLA: Math 32B, Winter 2017

Instructor: Noah White
Date: 27 February, 2017

- This exam has 5 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

bie
3E
3F

Question	Points	Score
1	9	7
2	8	8
3	7	4
4	5	5
5	11	10
Total:	40	34

+1

35

Here are some formulas that you may find useful as some point in the exam.

$$\int \cos^2 x \, dx = \frac{1}{2}(x + \cos x \sin x)$$

$$\int \sin^2 x \, dx = \frac{1}{2}(x - \cos x \sin x)$$

$$\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x$$

Spherical coordinates are given by

$$\begin{aligned} x(\rho, \theta, \phi) &= \rho \cos\theta \sin\phi \\ y(\rho, \theta, \phi) &= \rho \sin\theta \sin\phi \\ z(\rho, \theta, \phi) &= \rho \cos\phi \end{aligned}$$

The Jacobian for the change of coordinates is $J = \rho^2 \sin\phi$.

1. Let $\mathcal{E}$ be the solid region defined by

sphere radius $\sqrt{a}$

$$x^2 + y^2 + z^2 \le a, \quad x, y, z \ge 0,$$

for a fixed constant $a > 0$. Suppose the region has a constant mass density of $\delta(x, y, z) = 1$.

(a) (2 points) Express the total mass of $\mathcal{E}$ as an iterated integral.

spherical

$0 \le \rho \le \sqrt{a}$ $\quad 0 \le \theta = 2\pi, \quad 0 \le \phi \le \pi$

$M = \boxed{\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} 1 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$

1

(b) (2 points) Find the total mass of $\mathcal{E}$.

$\int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt{a}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$\int_0^{2\pi} \int_0^{\pi} \frac{1}{3} a^{3/2} \sin\phi \, d\phi \, d\theta$

$\int_0^{2\pi} -\frac{1}{3} a^{3/2} \cos\phi \Big]_0^{\pi} d\theta$

$\int_0^{2\pi} -\frac{1}{3} a^{3/2} (-1 - 1) \, d\theta$

$\int_0^{2\pi} + \frac{2}{3} a^{3/2} \, d\theta$

$\frac{2}{3} a^{3/2} (2\pi)$

$\frac{4\pi}{3} a^{3/2}$

2

ECF

(c) (3 points) Express the coordinates of the center of mass of $\mathcal{E}$ as an iterated triple integral.

$(x_{cm}, y_{cm}, z_{cm}) = \frac{1}{\iiint \delta(x,y,z)} \iiint (x,y,z)\, \delta(x,y,z)\, dV$

From (b) $M = \frac{4\pi}{3} a^{3/2}$

$\boxed{(x_{cm}, y_{cm}, z_{cm}) = \frac{3}{4\pi} a^{-3/2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt{a}} (x,y,z)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$

3 ECF

(d) (2 points) Find the z coordinate of the center of mass.

$z = \rho\cos\phi$ spherical

$z_{cm} = \frac{3}{4\pi} a^{-3/2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt{a}} z \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$= \frac{3}{4\pi} a^{-3/2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt{a}} \rho^3 \sin\phi \cos\phi \, d\rho\, d\phi\, d\theta$

1 ECF

~~We know by symmetry that $z_{cm} = 0$~~

~~since it is a sphere, center of mass is $(0,0,0)$ esp. since~~

~~$\delta(x,y,z) = 1$ = uniform~~

~~(think of a centroid)~~

2. Consider the helix C, given by the parameterisation

$$\mathbf{r}(t) = \left(\cos t, \sin t, \frac{1}{2\pi}t\right) \quad t \in [0, 4\pi],$$

so that C is oriented with the z coordinate increasing.

(a) (4 points) Compute the length of C.

$f(x,y,z) = 1$, $ds = \|\vec{r}\,'(t)\|\,dt$

length of $C = \int_C 1\,ds$

$= \int_a^b \|\vec{r}\,'(t)\|\,dt$

$= \int_0^{4\pi} \sqrt{1+\frac{1}{4\pi^2}}\,dt$

$= \sqrt{1+\frac{1}{4\pi^2}}\; t \Big]_0^{4\pi}$

$= 4\pi\sqrt{1+\frac{1}{4\pi^2}}$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, \frac{1}{2\pi} \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \frac{1}{4\pi^2}}$

$= \sqrt{1+\frac{1}{4\pi^2}}$

(b) (4 points) Compute the work done by the field

$$\mathbf{F}(x, y, z) = \langle z^2, 2yz^2, 2z(x + y^2) - e^z \rangle$$

on a particle constrained to move on the curve C.

so complicated

check conservative

curl $(\vec{F}) = 0$?

$\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x}$ ✓

$\frac{\partial F_2}{\partial z} = 4yz = \frac{\partial F_3}{\partial y}$

$\frac{\partial F_3}{\partial x} = 2z = \frac{\partial F_1}{\partial z}$ and simply connected so $\vec{F}$ conservative

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Find f

$\int z^2 \, dx = z^2 x + \alpha(y, z)$

$\int 2yz^2 \, dy = y^2 z^2 + \beta(x, z)$

$\int 2z(x+y^2) - e^z \, dz = z^2(x+y^2) - e^z + \gamma(x, y)$

$f(x,y,z) = z^2(x+y^2) - e^z$ $\quad \langle z^2, 2yz^2, 2z(x+y^2) - e^z \rangle$ ✓

$ff \ (0, 4\pi)$ $\quad r(t) = \langle \cos t, \sin t, \frac{1}{2\pi} t \rangle$

~~$f(1, 0, 0)$~~

$f(1,0,2) - f(1,0,0) = (4(1) - e^2) - (-1)$

$= 5 - e^2$

4/7

3. For this question consider the vector field

2

$$\mathbf{F}(x,y) = \frac{1}{r^2}\langle y(r^2-1), x(r^2+1)\rangle,$$

where $r = \sqrt{x^2+y^2}$. This vector field is defined everywhere apart from the origin.

(a) (4 points) Is **F** conservative on the domain described above? Justify your answer. Try polar

curl$(\vec{F}) = 0$?

✓ $\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x}$

— See if simply connected ?

y, x can be positive or negative, but $r = \sqrt{x^2+y^2}$

Since polar coordinates

$y = r \sin\theta$ $x = r\cos\theta$, we set only $r \geq 0$ since

because of the square root ...

so vector field $\boxed{\vec{F} \text{ not conservative on domain above}}$

1

(b) (1 point) Give a domain on which **F** is conservative.

$\vec{F}$ conservative everywhere where x, y must be greater or equal to 0.

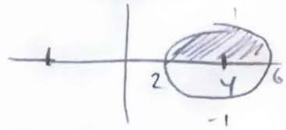
and except $(0,0)$

In other words $\vec{F}$ defined $x \geq 0$ AND $y \geq 0$ AND EXCLUDING $(0,0)$

1

(c) (2 points) Calculate the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$



where C is the ellipse $\frac{(x-4)^2}{2} + y^2 = 1$, oriented in the counter clockwise direction.

defined $0 \le \theta \le \pi$

$\vec{r}(t) = \langle 4 + \sqrt{2}\cos t, \sin t \rangle$

$\vec{r}\,'(t) = \langle -\sqrt{2}\sin t, \cos t \rangle \quad \|\vec{r}\,'(t)\| = \sqrt{\cos^2 t + 2\sin^2 t}$

path independent, since $\vec{F}$ regardless $\vec{r}$ is discarded.

$\int_C 1\, ds = \cancel{\int_0^1 \|\vec{r}\,'(t)\|\, dt}$

$= \int_0^{\pi} \sqrt{\cos^2 t + 2\sin^2 t}\; dt$

$= \frac{1}{2}(x + \cos x \sin x) + x - \cos x \sin x \Big]_0^{\pi}$

$= \frac{1}{2}(\pi + 0) + \pi - 0$

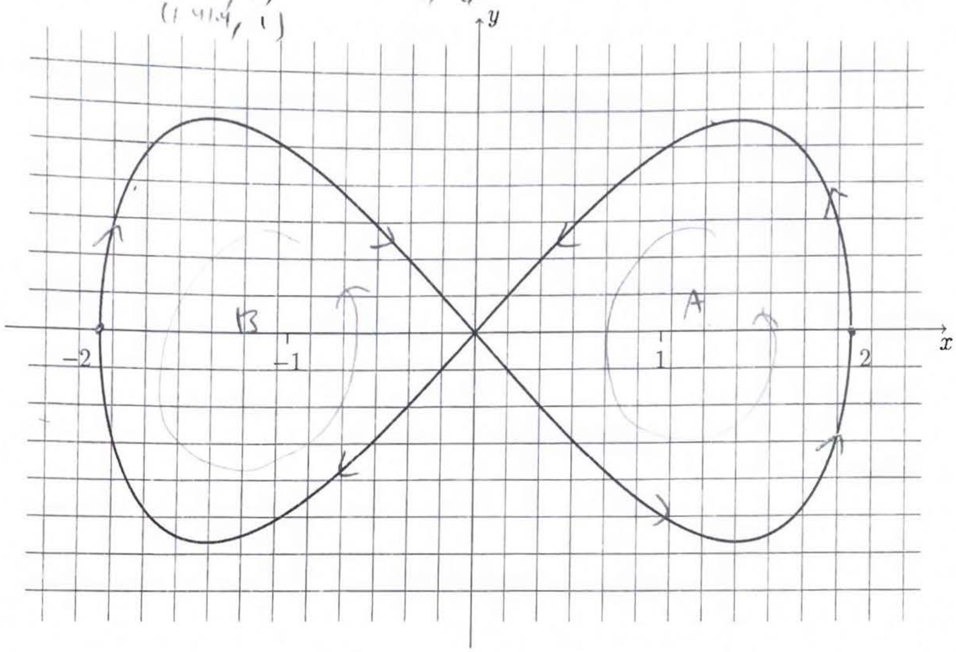
$= \frac{3}{2}\pi$

5

4. In this question assume that $\mathbf{E}$ is a vector field defined on the whole plane, apart from the points $(\pm 1, 0)$. The function $\mathbf{r}(t) = (2\cos t, \sin 2t)$ for $t \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ defines the curve C on the graph below

curl$(\vec{F}) = 0$ except there.

(-2, 0)
(1.414, 1)



1

4

(a) (1 point) Indicate on the above graph, the orientation of the curve. Drawn by arrows

(b) (4 points) Let $\mathcal{A}$ and $\mathcal{B}$ be the circles, radius $\frac{1}{2}$, and center $(1, 0)$ and $(-1, 0)$ respectively, both oriented counter clockwise. Suppose that

$$\int_{\mathcal{A}} \mathbf{E} \cdot \mathrm{dr} = 2 \quad \text{and} \quad \int_{\mathcal{B}} \mathbf{E} \cdot \mathrm{dr} = 1.$$

What is $\int_C \mathbf{E} \cdot \mathrm{dr}$? Justify your answer. see drawing

in defined domain curl$(\vec{F}) = 0$ so conservative, path independent

$$\int_C \vec{E} \cdot d\vec{r} = \int_A \vec{E} \cdot d\vec{r} + -\int_B \vec{E} \cdot d\vec{r}$$

negative since opposite to oriented curve

$$= 2 - 1$$

$$\boxed{\int_C \vec{E} \cdot d\vec{r} = 1}$$

5. The *hyperboloid* is Noah's favorite surface. It is given by the equation $x^2 + y^2 - z^2 = 1$.

(a) (3 points) Find a parameterisation

$$G(s,\theta) = (x(s,\theta), y(s,\theta), z(s,\theta)) \quad (s,\theta) \in \mathbb{R} \times [0, 2\pi]$$

for the hyperboloid. *Hint: Let* $z = s$.

$x^2+y^2 = 1+s^2$

r^2

$s\cos\theta \quad s\sin\theta$

$r = \sqrt{1+s^2}$

$G(s,\theta) = \left(\sqrt{1+s^2}\cos\theta, \sqrt{1+s^2}\sin\theta, s\right)$

3/3

Since $x^2+y^2 = 1+s^2$

$r^2 = 1+s^2$

$r = \sqrt{1+s^2}$

so $x = r\cos\theta = \sqrt{1+s^2}\cos\theta$

$y = r\sin\theta = \sqrt{1+s^2}\sin\theta$

(b) (5 points) Express the surface area of the hyperboloid between $z = a$ and $z = -a$ as an iterated integral.

$z = s$ so $-a \le s \le a$

$SA = \iint_S 1\, dS$ 5/5

$= \int_0^{2\pi}\int_{-a}^{a} \|\vec{N}(s,\theta)\|\, ds\, d\theta$

$= \int_0^{2\pi}\int_{-a}^{a} \sqrt{1+2s^2}\, ds\, d\theta$

$= \int_0^{2\pi}\int_{-a}^{a} \sqrt{1+2s^2}\, ds\, d\theta$

$dS = \|\vec{N}(u,v)\|\, du\, dv$

Find $\vec{N}$.

$G_s = \left(\frac{2s}{2}(1+s^2)^{-1/2}\cos\theta, \frac{2s}{2}(1+s^2)^{-1/2}\sin\theta, 1\right)$

$= \left(\frac{s}{\sqrt{1+s^2}}\cos\theta, \frac{s}{\sqrt{1+s^2}}\sin\theta, 1\right)$

$G_\theta = \left(-\sqrt{1+s^2}\sin\theta, \sqrt{1+s^2}\cos\theta, 0\right)$

$\vec{N} = G_s \times G_\theta \ldots$

$= \langle -\sqrt{1+s^2}\cos\theta, -\sqrt{1+s^2}\sin\theta, s\cos^2\theta + s\sin^2\theta \rangle$

$= \langle -\sqrt{1+s^2}\cos\theta, -\sqrt{1+s^2}\sin\theta, s \rangle$

$\|\vec{N}\| = \sqrt{1+s^2+s^2} = \sqrt{1+2s^2}$

(extra working room for part (b))

$1 + \tan^2\theta = \sec^2\theta$

(c) (3 points) Calculate the surface area.

$$SA = \int_0^{2\pi}\int_{-9}^{9} \sqrt{1+2r^2}\, dr\, d\theta$$

trig sub

$$= 2\int_0^{2\pi}\int_0^{9} \sqrt{1+2r^2}\, dr\, d\theta$$

since even, can do $\int_{-a}^{a}$ to $2\int_0^a$

let $r = \frac{1}{\sqrt{2}}\tan\theta$

$dr = \frac{1}{\sqrt{2}}\sec^2\theta$

$$= 4\pi \int_0^{9} \sqrt{1+2r^2}\, dr\, d\theta$$

$$= 4\pi \int_0^{\frac{1}{\sqrt{2}}\tan a} \frac{1}{\sqrt{2}}\sec^3\theta\, d\theta$$

$$= 8\pi$$

2/3