Midterm 2

UCLA: Math 32B, Spring 2018

Instructor: Noah White
Date: 21 May 2018

- This exam has 4 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

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Discussion section (please circle):

Day/TA	Ryan	Eli	Khang
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	9	9
2	12	12
3	9	7
4	10	10
Total:	40	38

6+6

Question 1 is multiple choice. Indicate your answers in the table below. *The following three pages will not be graded, your answers must be indicated here.*

Part	A	B	C	D
(a)				✓
(b)				✓
(c)	✓			
(d)	✓			
(e)				✓
(f)		✓		
(g)				✓
(h)	✓			
(i)		✓		

✓

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (1 point) If C is the straight line in $\mathbb{R}^2$ between $(0,0)$ and $(3,0)$, the integral $\int\int_C 1\, ds$ is equal to

?

A. -2

B. 2

C. -3

D. 3

$(0,0)$ $(3,0)$

$f(x) = 0$

$f(x,y) = 0x$

$0 \le x \le 3$

$y = 0$

$\int_0^3 1t\, dt$

$\vec{r}(t) = \langle t, 0 \rangle \quad 0 \le t \le 3$

$\frac{1}{2} t^2 \Big|_0^3$

$1 \|r'(t)\| \qquad r' = \langle 1, 0 \rangle \quad \sqrt{}\, t$

$\frac{9}{2}$

(b) (1 point) Consider the change of variables map $G(u,v) = (2v - u, u^2)$. The Jacobian of G is

A. $2u^2$

B. 0

C. -4

D. $-4u$

$$J = \det \begin{vmatrix} -1 & 2 \\ 2u & 0 \end{vmatrix}$$

$$= 0 - 4u = -4u$$

(c) (1 point) The divergence of the vector field $\mathbf{F} = \langle xy, -y^2 \rangle$ is

A. $-y$

B. x

C. $-x$

D. 0

$\mathrm{div}(F) = \nabla \cdot F = \partial_x F_1 + \partial_y F_2$

$y - 2y = -y$

(d) (1 point) The curl of the vector field $\mathbf{F} = \langle yz, xz, x\rangle$ is

A. $\langle -x, y-1, 0\rangle$
B. $\langle 0, -1, 0\rangle$
C. $\langle x, 0, z\rangle$
D. $\langle 0, 0, z-1\rangle$

$\nabla \times F$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ yz & xz & x \end{vmatrix}$$

$\hat{i}(0-x) - \hat{j}(1-y) + \hat{k}(z-z)$

$\langle -x, -1+y, 0\rangle$

(e) (1 point) The vector field $\mathbf{F} = \langle yz, zx, xy\rangle$ is

A. not conservative
B. conservative with potential function $x^2 + y^2 + z^2$.
C. conservative with potential function $x + y + z$
D. conservative with potential function xyz

xyz

ok (f) (1 point) Let C be the unit circle oriented in the clockwise direction and let $\mathbf{F} = \langle -e^{y^2-x}, 2ye^{y^2-x}\rangle$. The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is equal to

closed

A. π
B. 0
C. $-\pi$
D. 3

$x^2+y^2=r^2$ $\langle +r\cos\theta, r\sin\theta\rangle$

$r=1$ $\cancel{r}\sin^2\theta - r\cos\theta$ $\cancel{r}\sin\theta - r\cos\theta$

$-e$, $2r\sin\theta$

$0 \le \theta \le 2\pi$

t

$r'(t) = \langle -\sin t, \cos t\rangle$

conservative? $e^{y^2-x} = f$ ✓

(g) (1 point) Let $\mathbf{F} = \nabla(2xy + x^2)$ and let C be the curve given by the portion of the parabola $y = x(4-x)$ where $y \geq 0$, oriented from left to right. The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is equal to

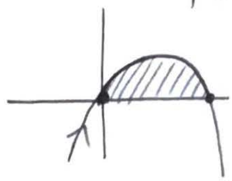
A. 1

B. 4

C. 9

D. 16

$f = \nabla F$



$F = \langle 2y + 2x, 2x \rangle$

$\vec{r}(t) = \langle t, 4t - t^2 \rangle$

$r(4) = \langle 4, 0 \rangle \quad r(0) = \langle 0, 0 \rangle$

✓ conservative

$F = f(4) - f(0)$

$f = 2xy + x^2 \qquad f(4) = 0 + 16$

$f(0) = 0 + 0 \qquad = \boxed{16}$

(h) (1 point) Let S be the portion of the unit sphere where $x \geq 0$ with normal vector pointing outwards. Let $\mathbf{F} = \langle 3, 0, 0 \rangle$, a constant vector field. The surface integral $\iint_S \vec{\mathbf{F}} \cdot d\mathbf{S}$ is

A. positive.

~~B.~~ zero.

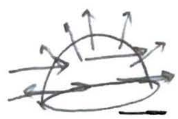
C. negative.

? D. not possible to tell without calculating.

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/2$

unit sphere $= (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \qquad \rho = 1$



$\iint \langle 3, 0, 0 \rangle \cdot$

$R^2 \sin\phi = \sin\phi \} \|N\|$ of sphere

(i) (1 point) Consider the parametrised surface given by $G(u, v) = (v^2 - u, u, uv)$. The normal vector at $G(1, 0) = (-1, 1, 0)$ is

A. $(1, 0, 0)$

B. $(1, 1, 0)$

C. $(1, 1, 1)$

D. $(0, 0, 0)$

$T_u = \langle -1, 1, v \rangle$

$T_v = \langle 2v, 0, u \rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & v \\ 2v & 0 & u \end{vmatrix}$$

$\hat{i}(u) - \hat{j}(-u - 2v^2) + \hat{k}(0 - 2v)$

$N = \langle u, 2v^2 + u, -2v \rangle$

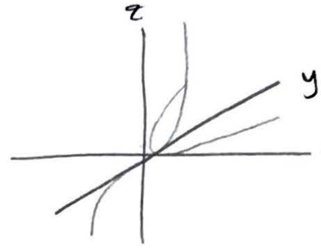
$N(1, 0) = \langle 1, 1, 0 \rangle$

2. (a) (6 points) The *twisted cubic* is the curve in $\mathbb{R}^3$ that is the intersection of the surfaces $y = x^2$ and $z = x^3$. Let C be the part of the twisted cubic where $x \in [0, 1]$. Let $f(x, y, z) = (1 + 4y + 9xz)^{-\frac{1}{2}}$. Evaluate the integral

$$\int_C f(x, y, z)\, ds$$ Scalar line integral

In your solution, to get full points, please carefully indicate a parametrisation of the curve, the single variable definite integral and the final solution. *Hint: If (x, y, z) is a point on the curve where $x = t$, then $y = t^2$ and $z = t^3$*

$x = t$
$y = t^2$ $\quad 0 \leq t \leq 1$
$z = t^3$



$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$0 \leq t \leq 1$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$

$f(x, y, z) = (1 + 4y + 9xz)^{-1/2}$

$f(\vec{r}(t)) = (1 + 4t^2 + 9(t)(t^3))^{-1/2}$

$= (1 + 4t^2 + 9t^4)^{-1/2}$

$\int_0^1 f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$

$\int_0^1 \frac{1}{\cancel{\sqrt{1 + 4t^2 + 9t^4}}} \cancel{\sqrt{1 + 4t^2 + 9t^4}}\, dt$

$\int_0^1 1\, dt = t\Big|_0^1 = \boxed{1}$

or

(b) (6 points) Let $\mathcal{D}$ be the region in $\mathbb{R}^2$ bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$ and $xy^2 = 2$. Find the area of $\mathcal{D}$. Give your answer as an exact value (logs, trig functions, exponentials and square roots are ok) *Hint: use a change of coordinates.*

$xy=1$

$xy=2$

$xy^2=1$

$xy^2=2$

$1 \leq \overbrace{xy}^{u} \leq 2$

$1 \leq \overbrace{xy^2} \leq 2$

$\text{area} = \iint 1 \, dA$ ★Jacobian

$x = Au + Cv \qquad y = Bu + Dv$

$u = xy$

$v = xy^2$

$J = \det \begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix}$

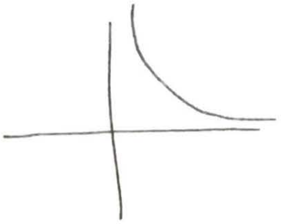
$y = \frac{1}{x} \qquad y = \sqrt{\frac{1}{x}}$

$\cancel{x} = A(\cancel{x}y) + C(\cancel{x}y^2) \qquad y = B(xy) + D(xy^2)$

$1 = Ay + Cy^2 \qquad 1 = Bx + Dxy$

$\frac{1}{x} = \sqrt{\frac{1}{x}}$

$\frac{1}{x^2} = \frac{1}{x}$



$\int_1^2 \int_1^2$

$x = x^2$

$0 = x^2 - x$

$0 = x(x-1)$

$x = 0, 1$

$u = x(1/x) = 1$

points of intersection:

~~[illegible]~~

↳ need to solve for x & y in terms of u & v

$u = xy \qquad \frac{u}{x} = y \qquad \frac{u}{u^2/v} = y$

$v = xy^2 \qquad \cancel{x} \cdot \frac{v}{u^2} = y \qquad \frac{v}{u} = y$

$v = x\left(\frac{u}{x}\right)^2$

$v = \frac{u^2}{x}$

$x = \frac{u^2}{v}$

$J = \det \begin{vmatrix} \frac{2u}{v} & \cancel{u^2 \ln v} \\ \cancel{v \ln u} & \frac{1}{u} \end{vmatrix}$

$J = \frac{2\cancel{u}}{v} \cdot \frac{1}{\cancel{u}} - u^2 \ln v \cdot v \ln u$

$= \frac{2}{v} - (u^2 \ln v)(v \ln u)$

See back of page →

$J = \frac{2}{v} - (u^2 \ln v)(v \ln u)$

$\int_1^2 \int_1^2 \frac{2}{v} - u^2 v \ln v \ln u \, du \, dv$

$J = \det \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{2\cancel{u}}{v} \cdot \frac{1}{\cancel{u}} - \left(\frac{-\cancel{u^2}}{v^{\cancel{2}}} \cdot \frac{-\cancel{v}}{\cancel{u^2}} \right)$

$\frac{2}{v} - \left(\frac{1}{v} \right) = \frac{1}{v}$

$x = u^2 v^{-1}$ $\quad y = v u^{-1}$

$\partial_u x = 2uv^{-1}$ $\quad \partial_u y = -vu^{-2}$

$\partial_v x = -u^2 v^{-2}$ $\quad \partial_v y = u^{-1}$

$\int_1^2 \int_1^2 \frac{1}{v} \, dv \, du$

$\int_1^2 \ln v \Big|_1^2 \, du$

$\int_1^2 \ln 2 - \cancel{\ln 1}^{\,0} \, du$

$\int_1^2 \underbrace{\ln 2}_{\text{constant}} \, du$

$(\ln 2) u \Big|_1^2$

$\boxed{2 \ln 2 - \ln 2}$

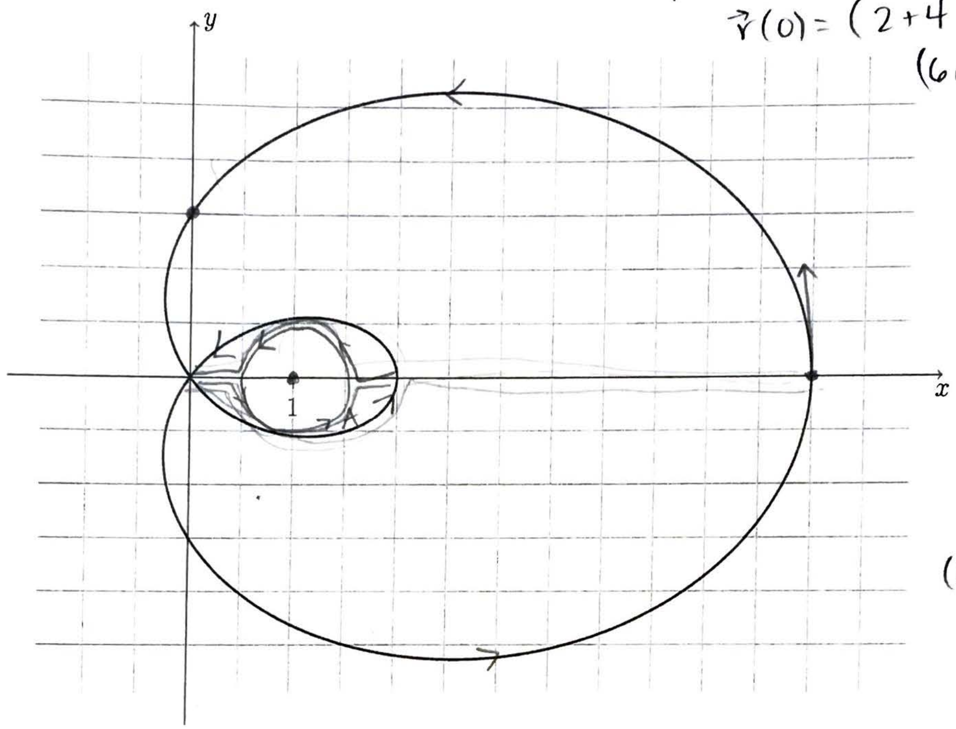
$\boxed{= \ln 2}$

3. In this question assume that **E** is a vector field defined on the whole of $\mathbb{R}^2$, apart from the points $(\pm 1, 0)$. Suppose that $\nabla \times \mathbf{E} = 0$. The function $\mathbf{r}(t) = (2\cos t + 4\cos^2 t, 2\sin t + 4\cos t \sin t)$ for $t \in [-\frac{2\pi}{3}, \frac{4\pi}{3}]$ traces out the curve C on the graph below. To give you an idea: it starts at the origin, traces out the large loop, returns to the origin when $t = 2\pi/3$, then traces out the small loop and then returns to the origin once more.

$\vec{r}(t) = (\cancel{2\cos 0 + 4}$

$\vec{r}(0) = (2+4, 0)$

$(6,0)$



$\vec{r}(\pi/2) =$

$(0+0, 2)$

$(0,2)$

$(6,0) \to (0,2)$

$0 \to \pi/2$

$(6,0) \to (0,2)$

(a) (2 points) Indicate on the above graph, the orientation of the curve. *Hint: calculating some tangent vectors might help.* 2 See above

(b) (7 points) Let $\mathcal{A}$ be the circle radius $\frac{1}{2}$, and centre $(1, 0)$ (so it fits entirely within the small loop above) oriented counter clockwise. Suppose that

$$\int_{\mathcal{A}} \mathbf{E} \cdot d\mathbf{r} = 2$$

$r'(t) =$

What is $\int_C \mathbf{E} \cdot d\mathbf{r}$? Justify your answer.

loops around circle twice in the same orientation, so multiply 2 by how many loops

$2 * 2 = \boxed{4}$ (straight line portions cancel)

conservative?

Show more formally.

5

4. Consider the surface S given by the equation $x^2 + y^2 = (z^2 - z + 1)^2$ where $0 \le z \le 1$.

(a) (4 points) Parametrise S in the form $G(\theta, z) = (x(\theta, z), y(\theta, z), z)$.

$x^2+y^2 = \underbrace{(z^2-z+1)^2}$ $\quad 0 \le z \le 1$ cylindrical

$\langle \overset{r}{z}\cos\theta, \overset{r}{z}\sin\theta, z\rangle$

$r = z^2 - z + 1$

para $= \langle (z^2-z+1)\cos\theta, (z^2-z+1)\sin\theta, z\rangle$

$0 \le \theta \le 2\pi$

$0 \le z \le 1$

(b) (6 points) Find $\iint_S \mathbf{E} \cdot d\mathbf{S}$, where $\mathbf{E} = \langle \frac{x}{(z^2-z+1)^2}, \frac{y}{(z^2-z+1)^2}, 0\rangle$.

Vector Surface integral

turn over ↳ next page

$\vec{E} = \left\langle \frac{x}{(z^2-z+1)^2}, \frac{y}{(z^2-z+1)^2}, 0 \right\rangle$

$E(G(z,\theta)) = \left\langle \frac{\cancel{(z^2-z+1)}\cos\theta}{(z^2-z+1)^{\cancel{2}}}, \frac{\cancel{(z^2-z+1)}\sin\theta}{(z^2-z+1)^{\cancel{2}}}, 0 \right\rangle$

$= \left\langle \frac{\cos\theta}{z^2-z+1}, \frac{\sin\theta}{z^2-z+1}, 0 \right\rangle$

$T_\theta = \langle -(z^2-z+1)\sin\theta, (z^2-z+1)\cos\theta, 0\rangle$

$T_z = \langle \cos\theta(2z-1), \sin\theta(2z-1), 1\rangle$

$N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ & & \\ & & \end{vmatrix}$

$N = \hat{i}\left((z^2-z+1)\cos\theta\right) - \hat{j}\left(-(z^2-z+1)\sin\theta\right)$
$+ \hat{k}\left(-\sin^2\theta(2z-1)(z^2-z+1) - \cos^2\theta(2z-1)(z^2-z+1)\right)$

$N = \langle (z^2-z+1)\cos\theta, (z^2-z+1)\sin\theta, -(2z-1)(z^2-z+$

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4b) cont'd

$$\|N\| = \sqrt{(z^2-z+1)^2\cos^2\theta + (z^2-z+1)^2\sin^2\theta + ((2z-1)(z^2-z+1))^2}$$

$$= \sqrt{(z^2-z+1)^2 + (2z-1)^2(z^2-z+1)^2}$$

$$= \sqrt{(z^2-z+1)^2\left[1 + (2z-1)^2\right]}$$

$$= (z^2-z+1)\sqrt{1+4z^2-4z+1}$$

$$= (z^2-z+1)\sqrt{4z^2-4z+2}$$

$$\int_0^{2\pi}\int_0^1 \frac{\cos^2\theta}{\cancel{z^2-z+1}} + \frac{\sin^2\theta}{\cancel{z^2-z+1}} + 0 \, dz\, d\theta$$

$$\int_0^{2\pi}\int_0^1 1 \, dz\, d\theta$$

$$\int_0^{2\pi} 1 \, d\theta = \boxed{2\pi}$$