1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (1 point) If $\mathcal{R} = [-1, 0] \times [2, 6]$, the integral $\iint_{\mathcal{R}} \frac{1}{2}\, dA$ is equal to

A. 2

B. 0

C. 5

D. 4

1 4

$\frac{1}{2} \times 1 \times 4$

$\frac{1}{2}x$

$0 - - \frac{1}{2}x$

$\frac{1}{2}(4)$

(b) (1 point) If $\mathcal{R} = [0, 1] \times [0, 1]$, the integral $\iint_{\mathcal{R}} 4xy\, dA$ is equal to

A. −1

B. 4

C. −4

D. 1

1 1

$\frac{4x^2}{2}y$

$2x^2y\Big|_0^1$

$2x^2y\Big|_0^1$

$2y$

$2y \quad y^2$

$y^2\Big|_0^1$

2 2 1 1.

(c) (1 point) If $\mathcal{B} = [-1, 1] \times [0, 1] \times [3, 4]$, the integral $\iiint_{\mathcal{B}} -2\, dV$ is equal to

A. −4

B. 1

C. −2

D. 2

2 1 1

$2x - 2$

(d) (1 point) If $\mathcal{R} = [-2, 2] \times [3, 6]$, the integral $\iint_{\mathcal{R}} xe^{x^2+y^2}\,dA$ is equal to

A. 0

B. 2

C. -1

D. $3\pi^2$

$\frac{1}{2}\iint e^u\,du$

0

$\frac{1}{2}e^{4+y^2} - \frac{1}{2}e^{4+y^2}$ $\quad \frac{1}{2}e^{x^2+y^2}\Big|_{-2}^{2}$

(e) (1 point) If $\mathcal{B} = [0, 1] \times [0, 3] \times [0, 3]$, the integral $\iiint_{\mathcal{B}} 2x\,dV$ is equal to

A. 3

B. 18

C. 1

D. 9

Hint: integrate in the order $dx\,dy\,dz$

$1x^2$

$x^2\Big|_0^1$

1

(f) (1 point) The Jacobian of the change of coordinates $G(u, v) = (u^2 + v, v^2 + u)$

A. $uv + 1$

B. $4uv - 1$

C. $2v^2 - 1$

D. $4u^2v^2$

$2u \cdot 2v - 1 \cdot 1$ $\quad 2u \cdot 2v - 1 \cdot 1$

$4uv - 1$

(g) (1 point) If $\mathcal{D}$ is the region $4 \le x^2 + y^2 \le 16$, where $y \ge 0$ then after changing to polar coordinates, the integral $\iint_{\mathcal{D}} x \, dA$ becomes

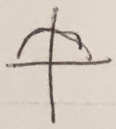
~~A.~~ $\int_0^{\pi} \int_2^3 r\cos\theta \, dr \, d\theta$

~~B.~~ $\int_0^{2\pi} \int_2^4 r^2 \sin\theta \, dr \, d\theta$

~~C.~~ $\int_0^{\pi} \int_2^4 r^3 \sin 2\theta \, dr \, d\theta$

D. $\int_0^{\pi} \int_2^4 r^2 \cos\theta \, dr \, d\theta$

$\int_0^{\pi} \int_2^4 r^2 \cos\theta$



$2r$

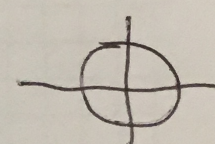
(h) (1 point) The integral of $2\sqrt{x^2+y^2}$ over the disc $x^2 + y^2 \le 1$ is

A. $\frac{2\pi}{3}$

B. 2π

C. $\frac{4\pi}{3}$

D. π



$\int_0$

4π

$\frac{2r^3}{3}$ $\int_0^{2\pi}\int_0^1 2r^2$ $\frac{2r^3}{3}\Big|_0^1$ $\frac{2}{3}\cdot 2\pi$

4

$\frac{2}{3}$ 2π

0.25

π^2 0.5, $\frac{1}{2}$

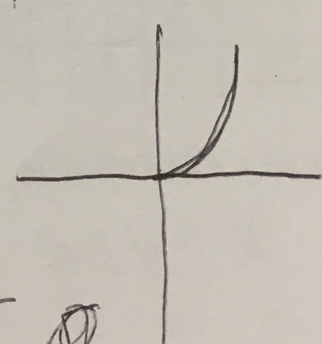
(i) (1 point) If $\mathcal{D}$ is the region between the curves $y = x^2$ and $y = \sin(\frac{1}{2}\pi x)$ in the first quadrant then $\mathcal{D}$ has the description

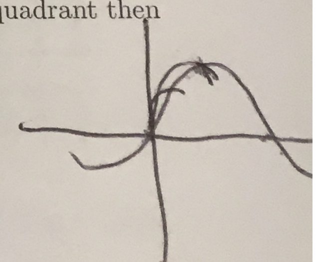
~~A.~~ $0 \le x \le \pi, \ \sin(\frac{1}{2}\pi x) \le y \le x^2$

B. $0 \le x \le 1, \ \sin(\frac{1}{2}\pi x) \le y \le x^2$

~~C.~~ $0 \le x \le \pi, \ 0 \le y \le \sin(\frac{1}{2}\pi x)$

D. $0 \le x \le 1, \ x^2 \le y \le \sin(\frac{1}{2}\pi x)$





$\frac{\pi}{4}$

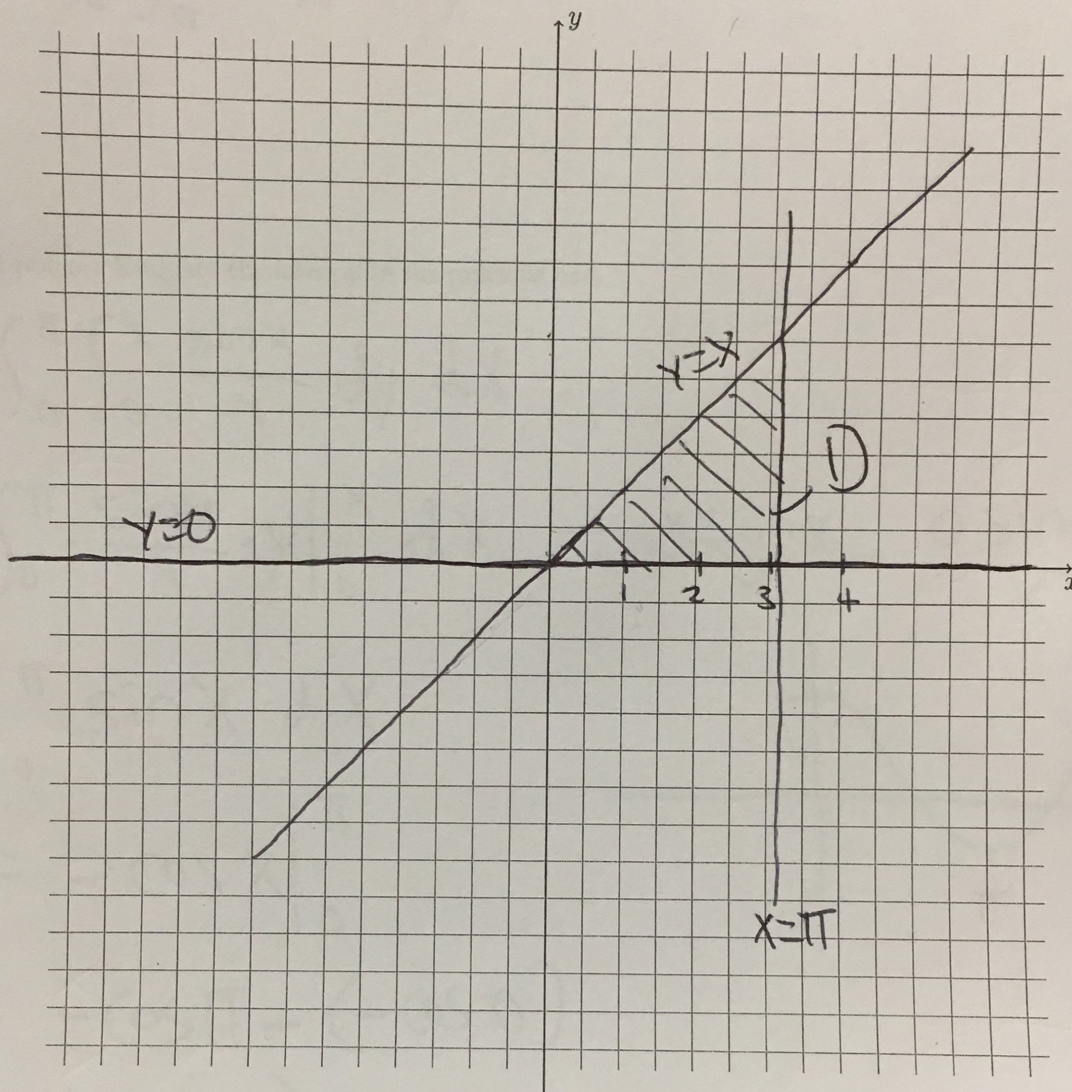
D. $\frac{1}{4}$ $\frac{\sqrt{2}}{2}$, $\frac{\phi}{2}$

$\frac{2\sqrt{2}}{4}$

2. In this question we will consider the region $\mathcal{D}$ which bounded by the lines
   - $y = 0$,
   - $y = x$, and
   - $x = \pi$.

   (a) (2 points) Sketch the region $\mathcal{D}$ on the graph provided.

   2



   (b) (1 point) Express $\mathcal{D}$ as a vertically simple region, i.e. in the form $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$.

   1

   $0 \leq X \leq \pi \quad 0 \leq Y \leq X$

   (c) (1 point) Express $\mathcal{D}$ as a horizontally simple region, i.e. in the form $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$.

   1

   $0 \leq y \leq \pi \quad y \leq X \leq \pi$

(d) (2 points) Write the integral

$$\iint_D \frac{\sin x}{x}\, dA$$

as an iterated integral (in either order is fine)

✓ 2

$$\int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\, dy\, dx$$

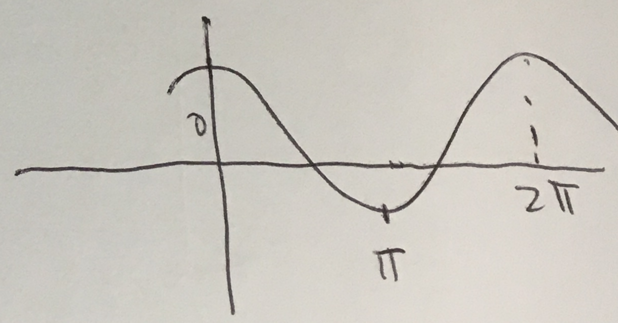
(e) (4 points) Evaluate the integral in the previous part.

4

$$\int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^{\pi} \frac{\sin x}{x} \cdot y\Big|_0^{x}\, dx \Rightarrow \frac{x\sin x}{x} - \frac{0 \sin x}{x}$$

✓ $$= \int_0^{\pi} \sin x\, dx$$



$$= -\cos x\Big|_0^{\pi}$$

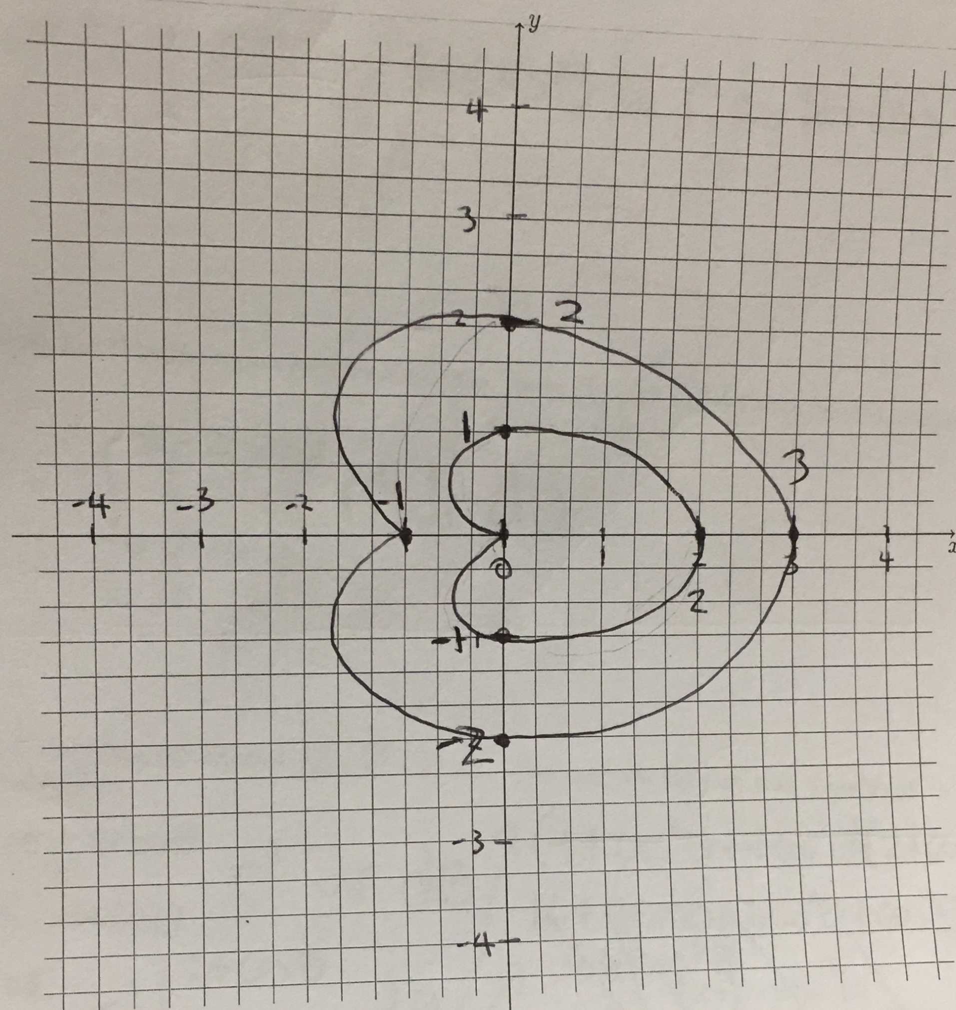
$$= -\cos\pi - (-\cos 0)$$

$$= 1 - (-1)$$

$$= 2$$

3. In this question, consider the curves $r = 1 + \cos\theta$ and $r = 2 + \cos\theta$.

   (a) (4 points) Sketch both the curves on the graph provided. Make sure to indicate where the curves cross the axes.



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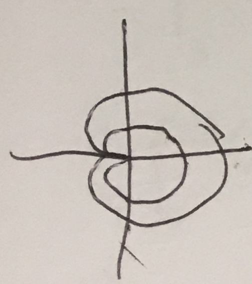
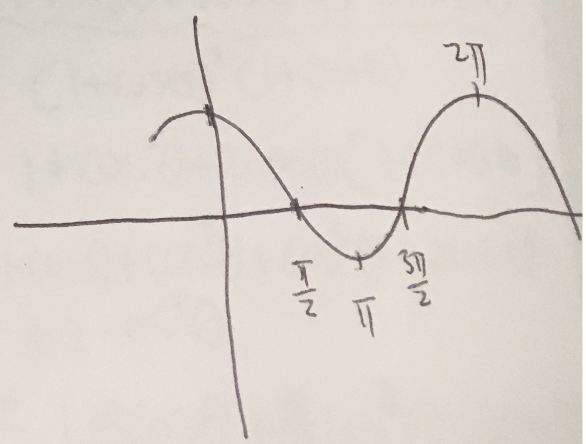




$$r^2 = r + r\cos\theta$$

$$x^2 + y^2 = \sqrt{x^2+y^2} + x$$

(b) (2 points) Write the region *between* the two curves as a radially simple region, i.e in the form $\varphi \le \theta \le \psi$ and $r_1(\theta) \le r \le r_2(\theta)$ for some functions r_1 and r_2.

$0 \le \theta \le 2\pi \qquad 1+\cos\theta \le r \le 2+\cos\theta$

2

(c) (2 points) Let $\mathcal{D}$ be the region between the curves. Write $\iint_{\mathcal{D}} \sqrt{x^2+y^2}\, dA$ as an iterated integral.

r

$$\int_0^{2\pi}\int_{1+\cos\theta}^{2+\cos\theta} r^2\, dr\, d\theta$$

✓

(d) (4 points) Calculate the integral $\iint_{\mathcal{D}} \sqrt{x^2+y^2}\, dA$. You may use the fact that $\int \cos^2\theta\, d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$.

$$\int_0^{2\pi}\int_{1+\cos\theta}^{2+\cos\theta} r^2\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\Big|_{1+\cos\theta}^{2+\cos\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(2+\cos\theta)^3}{3} - \frac{(1+\cos\theta)^3}{3}\, d\theta$$

$(4+\cos^2\theta+4\cos^2\theta)(2+\cos\theta)$

$16+\cancel{4\cos\theta}+2\cos^2\theta+\cos^3\theta+8\cos^2\theta$

$\cancel{4}\,4\cos^3\theta$

$\cancel{(2+\cos\theta)^3 =}$

$\cancel{16+4\cos\theta+10\cos^2\theta+5\cos^3\theta}$

$(1+\cos\theta)^2(1+\cos\theta)$

$1+\cos^2\theta+2\cos\theta(1+\cos\theta)$

$1+\cos\theta+\cos^2\theta+\cos^3\theta+2\cos\theta$

$+2\cos^3\theta$

$= 1+3\cos\theta+\cancel{4}\cos^3\theta+3\cos^3\theta$

$(2+\cos\theta)^3$

$(4+\cos^2\theta+4\cos\theta)(2+\cos\theta)$

$8+4\cos\theta+2\cos^2\theta+\cos^3\theta+8\cos\theta+4\cos^3\theta$

$8+12\cos\theta+2\cos^2\theta+5\cos^3\theta$

$8+12\cos\theta+2\cos^2\theta+5\cos^3\theta-1-3\cos\theta-\cos^2\theta-3\cos^3\theta$

$= 7+9\cos\theta+\cos^2\theta+2\cos^3\theta$

next

$$\int_0^{2\pi} 7 + 9\cos\theta + \cos^2\theta + 2\cos^3\theta \; d\theta$$

$$14\pi + \frac{2\pi}{2} + \cancel{\frac{\sin(4\pi)}{4}} \cancel{+ \sin\pi} \qquad \int \cos^2\theta \, d\theta = \frac{x}{2} + \frac{\sin 2x}{4}$$

$$= 15\pi \qquad \int \cos^3\theta \, d\theta = \sin x - \frac{\sin^3 x}{3} + C$$

3

$$\therefore \int_0^{2\pi} \cos^2\theta \, d\theta = \frac{2\pi}{2}$$

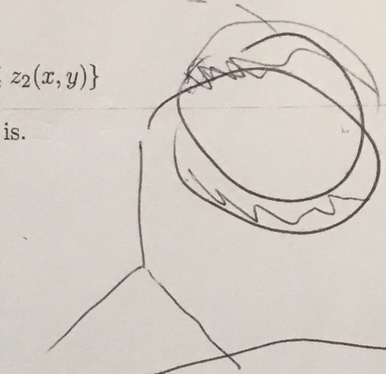
$$2\int_0^{2\pi} \cos^3\theta \, d\theta = 0$$

4. Consider the region $\mathcal{E}$ in the intersection of the two balls $x^2+y^2+(z-\frac{1}{2})^2 \le 1$ and $x^2+y^2+(z+\frac{1}{2})^2 \le 1$.

(a) (4 points) Describe the region in the form

$$\mathcal{E} = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \mathcal{D},\ z_1(x,y) \le z \le z_2(x,y)\}$$

for $\mathcal{D}$ a region in the xy-plane. Your answer should specify what $\mathcal{D}$ is.



$$x^2+y^2+(z+\tfrac{1}{2})^2 \le 1$$
$$(z+\tfrac{1}{2})^2 = 1-x^2-y^2$$
$$z+\tfrac{1}{2} = \sqrt{1-x^2-y^2}$$
$$z = \sqrt{1-x^2-y^2} - \tfrac{1}{2}$$
$$x^2+y^2+(z-\tfrac{1}{2})^2 = 1$$
$$z = \sqrt{1-x^2-y^2} + \tfrac{1}{2}$$

$$\boxed{-\sqrt{1-x^2-y^2}+\tfrac{1}{2} \le z \le \sqrt{1-x^2-y^2} - \tfrac{1}{2}}$$

$$x^2+y^2+\tfrac{1}{4} = 1$$
$$x^2+y^2 = \tfrac{3}{4}$$

$$\boxed{0 \le x \le \tfrac{\sqrt{3}}{2}}$$

$$\boxed{-\sqrt{\tfrac{3}{4}-x^2} \le y \le \sqrt{\tfrac{3}{4}-x^2}}$$

2/4

(b) (5 points) Compute the volume of the region $\mathcal{E}$.

$$\int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2-y^2} - \tfrac{1}{2} + \sqrt{1-x^2-y^2} - \tfrac{1}{2}\, dz\, r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}} r\, dr\, d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{\frac{\sqrt{3}}{2}} d\theta$$

$$= \int_0^{2\pi} \frac{3}{4}\cdot\frac{1}{2}\, d\theta$$

$$= \int_0^{2\pi} \frac{3}{8}\, d\theta$$

$$= \frac{3}{\cancel{8}\,4}\cdot\cancel{2}\pi$$

$$= \boxed{\frac{3\pi}{4}}$$

4/5