Discussion section (please circle):

Day/TA	Gyu Eun	Ben	Robbie
Tuesday	3A	3C	3E
Thursday	3B	3D	3F

Question	Points	Score
1	9	9
2	10	10
3	12	12
4	9	8
Total:	40	39

6+6

Question 1 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)	X			
(b)				X
(c)	X			
(d)	X			
(e)				X
(f)		X		
(g)				X
(h)			X	
(i)				X

A
D
A
A
D
B
D
C
D

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (1 point) If $\mathcal{R} = [-1, 0] \times [2, 6]$, the integral $\iint_{\mathcal{R}} \frac{1}{2}\, dA$ is equal to

A. 2
B. 0
C. 5
D. 4

$0 - -1 = 1$

$6 - 2 = 4$

$1 \cdot 4 \cdot \frac{1}{2} = 2$

A

(b) (1 point) If $\mathcal{R} = [0, 1] \times [0, 1]$, the integral $\iint_{\mathcal{R}} 4xy\, dA$ is equal to

A. -1
B. 4
C. -4
D. 1

$\int_0^1 \int_0^1 4xy\, dy\, dx$

$\int_0^1 \left[2xy^2\right]_0^1 dx = \int_0^1 2x\, dx = \left[x^2\right]_0^1 = 1$

D

(c) (1 point) If $\mathcal{B} = [-1, 1] \times [0, 1] \times [3, 4]$, the integral $\iiint_{\mathcal{B}} -2\, dV$ is equal to

A. -4
B. 1
C. -2
D. 2

$1 - -1 = 2$

$1 - 0 = 1$

$4 - 3 = 1$

$2 \cdot 1 \cdot 1 \cdot -2 = -4$

A

(d) (1 point) If $\mathcal{R} = [-2,2] \times [3,6]$, the integral $\iint_{\mathcal{R}} xe^{x^2+y^2}\, dA$ is equal to

A. 0

B. 2

C. −1

D. $3\pi^2$

$$\int_3^6 \int_{-2}^{2} x e^{x^2+y^2} \, dx\, dy$$

A

$$\int_3^6 \left[\frac{1}{2} e^{x^2+y^2}\right]_{-2}^{2} dy$$

$$\int_3^6 \frac{1}{2} e^{4+y^2} - \frac{1}{2} e^{4+y^2} \, dy = 0$$

(e) (1 point) If $\mathcal{B} = [0,1] \times [0,3] \times [0,3]$, the integral $\iiint_{\mathcal{B}} 2x\, dV$ is equal to

A. 3

B. 18

C. 1

D. 9

Hint: integrate in the order $dx\, dy\, dz$

D

$$\int_0^3 \int_0^3 \int_0^1 2x \, dx\, dy\, dz$$

9

$$\int_0^3 \int_0^3 \left[x^2\right]_0^1 dy\, dz = \int_0^3 \int_0^3 dy\, dz \qquad \int_0^3 3\, dz$$

(f) (1 point) The Jacobian of the change of coordinates $G(u,v) = (u^2+v, v^2+u)$

A. $uv + 1$

B. $4uv - 1$

C. $2v^2 - 1$

D. $4u^2v^2$

B

$$\frac{\partial x}{\partial u} = \begin{vmatrix} 2u & 1 \\ 1 & 2v \end{vmatrix} = \frac{\partial x}{\partial v}$$

$$\frac{\partial y}{\partial u} = \qquad = \frac{\partial y}{\partial v}$$

$4uv - 1$

(g) (1 point) If D is the region $4 \le x^2 + y^2 \le 16$, where $y \ge 0$ then after changing to polar coordinates, the integral $\iint_D x \, dA$ becomes

~~A. $\int_0^{\pi} \int_2^{3} r\cos\theta \, dr \, d\theta$~~

B. $\int_0^{2\pi} \int_2^{4} r^2 \sin\theta \, dr \, d\theta$

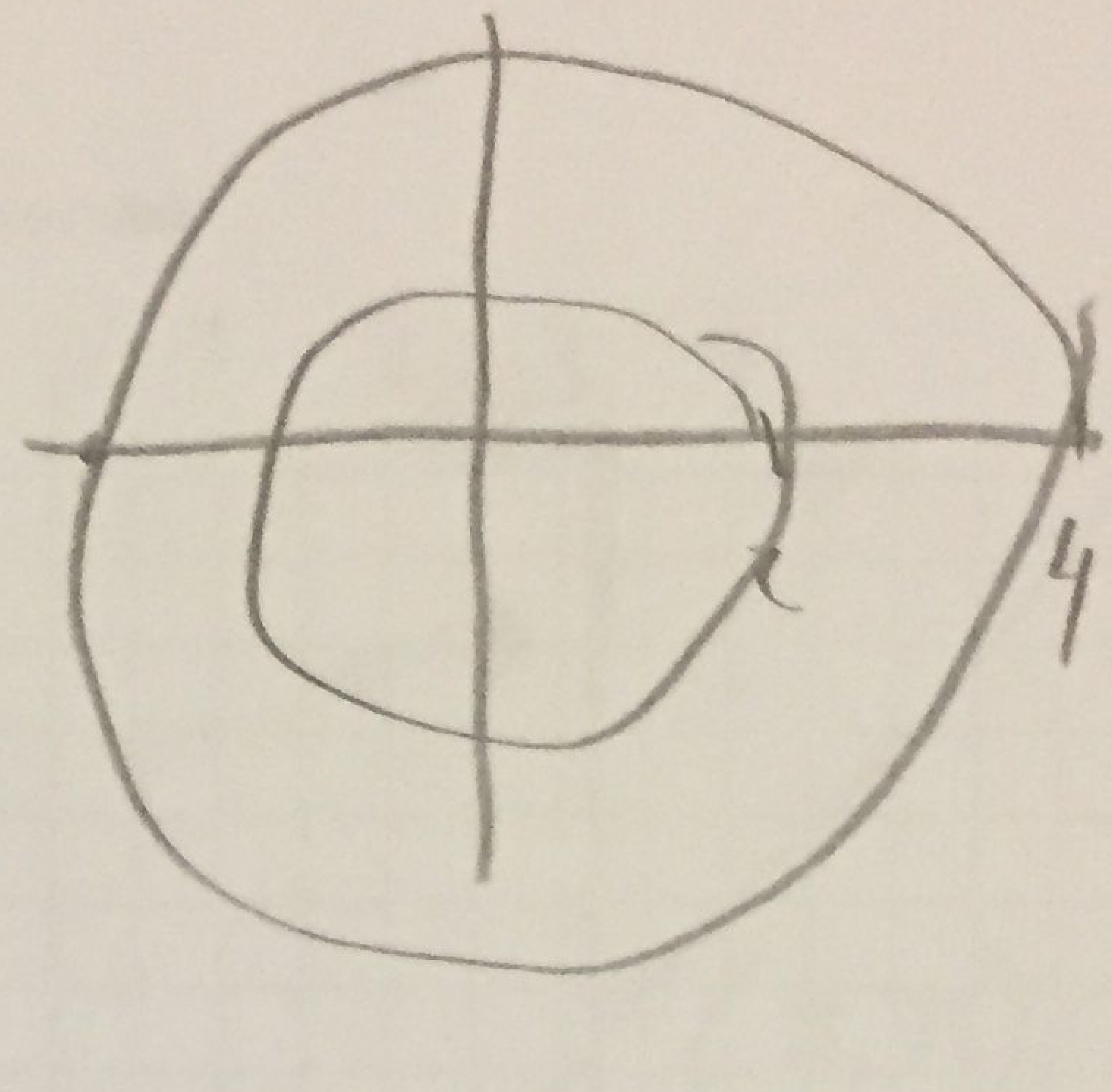
C. $\int_0^{\pi} \int_2^{4} r^3 \sin 2\theta \, dr \, d\theta$

D. $\int_0^{\pi} \int_2^{4} r^2 \cos\theta \, dr \, d\theta$

D

$x = r\cos\theta$

$r^2 \cos\theta \, dr \, d\theta$



(h) (1 point) The integral of $2\sqrt{x^2+y^2}$ over the disc $x^2+y^2 \le 1$ is

A. $\frac{2\pi}{3}$

B. 2π

C. $\frac{4\pi}{3}$

D. π

C

$$\int_0^{2\pi} \int_0^{1} 2r^2 \, dr \, d\theta$$

$$\int_0^{2\pi} \left[\frac{2}{3} r^3\right]_0^1 d\theta = \int_0^{2\pi} \frac{2}{3} \, d\theta = \frac{4\pi}{3}$$

(i) (1 point) If D is the region between the curves $y = x^2$ and $y = \sin(\frac{1}{2}\pi x)$ in the first quadrant then D has the description

~~A. $0 \le x \le \pi, \ \sin(\frac{1}{2}\pi x) \le y \le x^2$~~

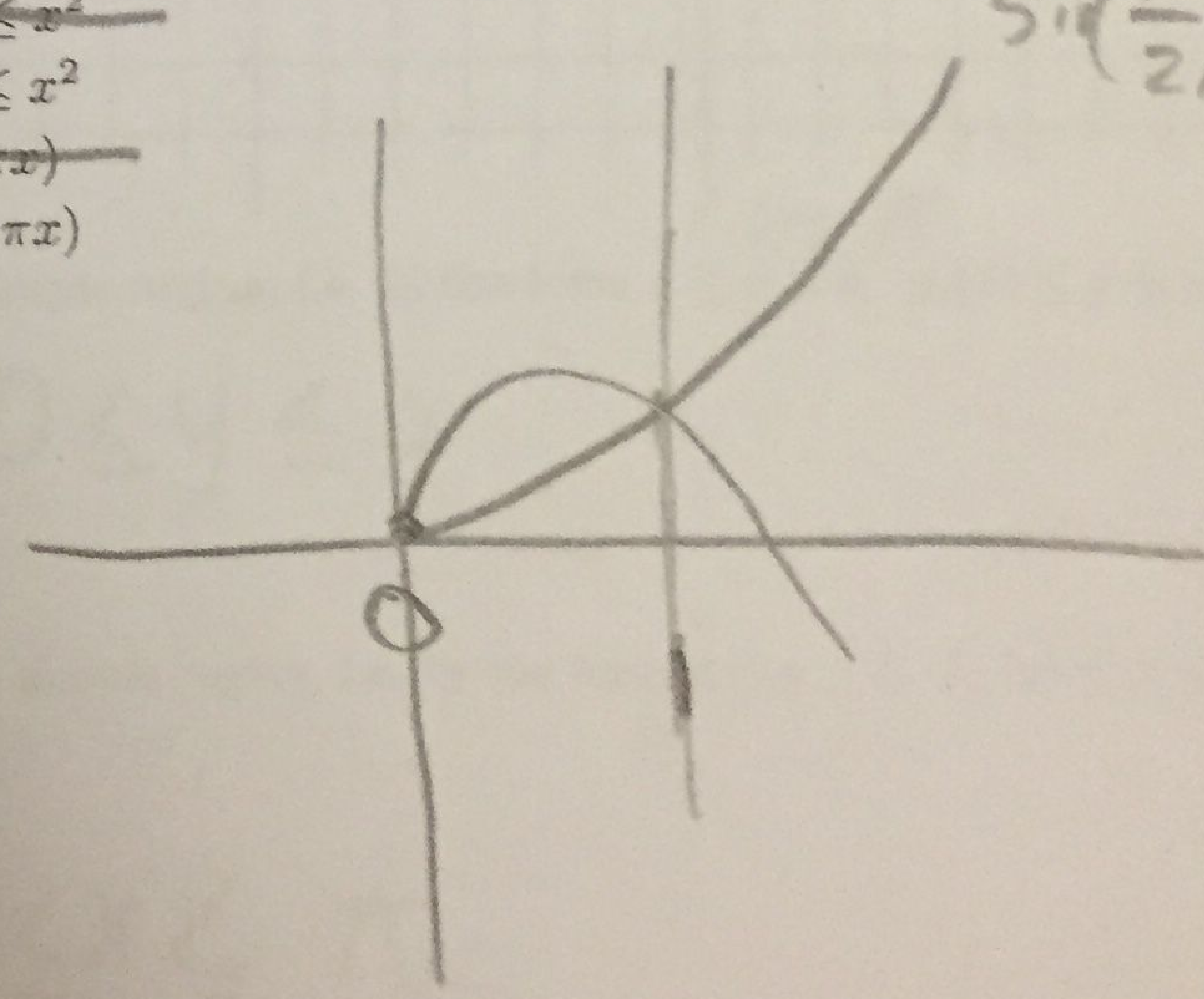
B. $0 \le x \le 1, \ \sin(\frac{1}{2}\pi x) \le y \le x^2$

~~C. $0 \le x \le \pi, \ 0 \le y \le \sin(\frac{1}{2}\pi x)$~~

D. $0 \le x \le 1, \ x^2 \le y \le \sin(\frac{1}{2}\pi x)$

D

$\sin\left(\frac{\pi}{2}\right) = 1$

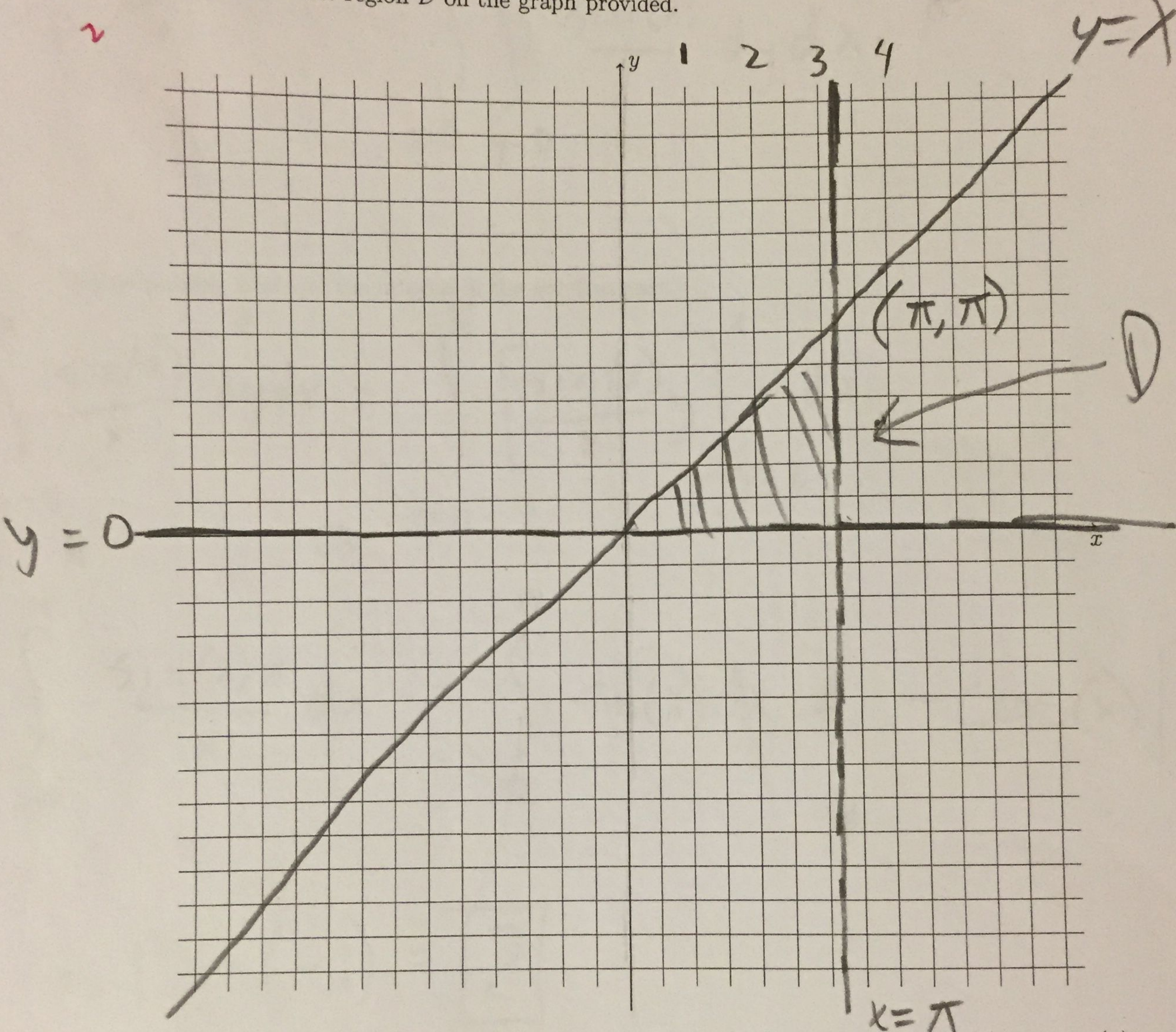


2. In this question we will consider the region $\mathcal{D}$ which bounded by the lines

- $y = 0$,
- $y = x$, and
- $x = \pi$.

(a) (2 points) Sketch the region $\mathcal{D}$ on the graph provided.

2



(b) (1 point) Express $\mathcal{D}$ as a vertically simple region, i.e. in the form $a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)$.

1 $0 \leq x \leq \pi,\ 0 \leq y \leq x$

(c) (1 point) Express $\mathcal{D}$ as a horizontally simple region, i.e. in the form $c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)$.

1

$0 \leq y \leq \pi,\ y \leq x \leq \pi$

(d) (2 points) Write the integral

2

$$\iint_D \frac{\sin x}{x}\, dA$$

as an iterated integral (in either order is fine)

$$\iint_D \frac{\sin x}{x}\, dA = \int_{x=0}^{x=\pi}\int_{y=0}^{y=x} \frac{\sin(x)}{x}\, dy\, dx$$

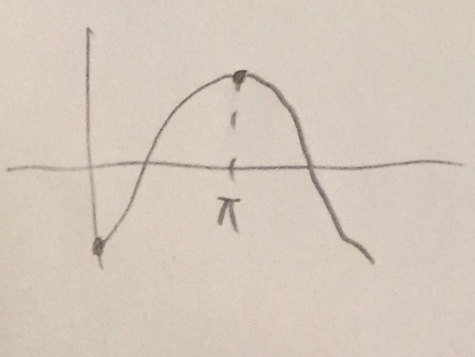
(e) (4 points) Evaluate the integral in the previous part.

4

$$\int_{x=0}^{\pi}\int_{y=0}^{x} \frac{\sin(x)}{x}\, dy\, dx = \int_{x=0}^{\pi} \left[\frac{\sin(x)}{x} y\right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{\pi} \frac{\sin(x)\, x}{x}\, dx = \int_0^{\pi} \sin(x)\, dx = -\cos(x)\Big|_0^{\pi}$$

$$= 1 - (-1) = \boxed{2}$$



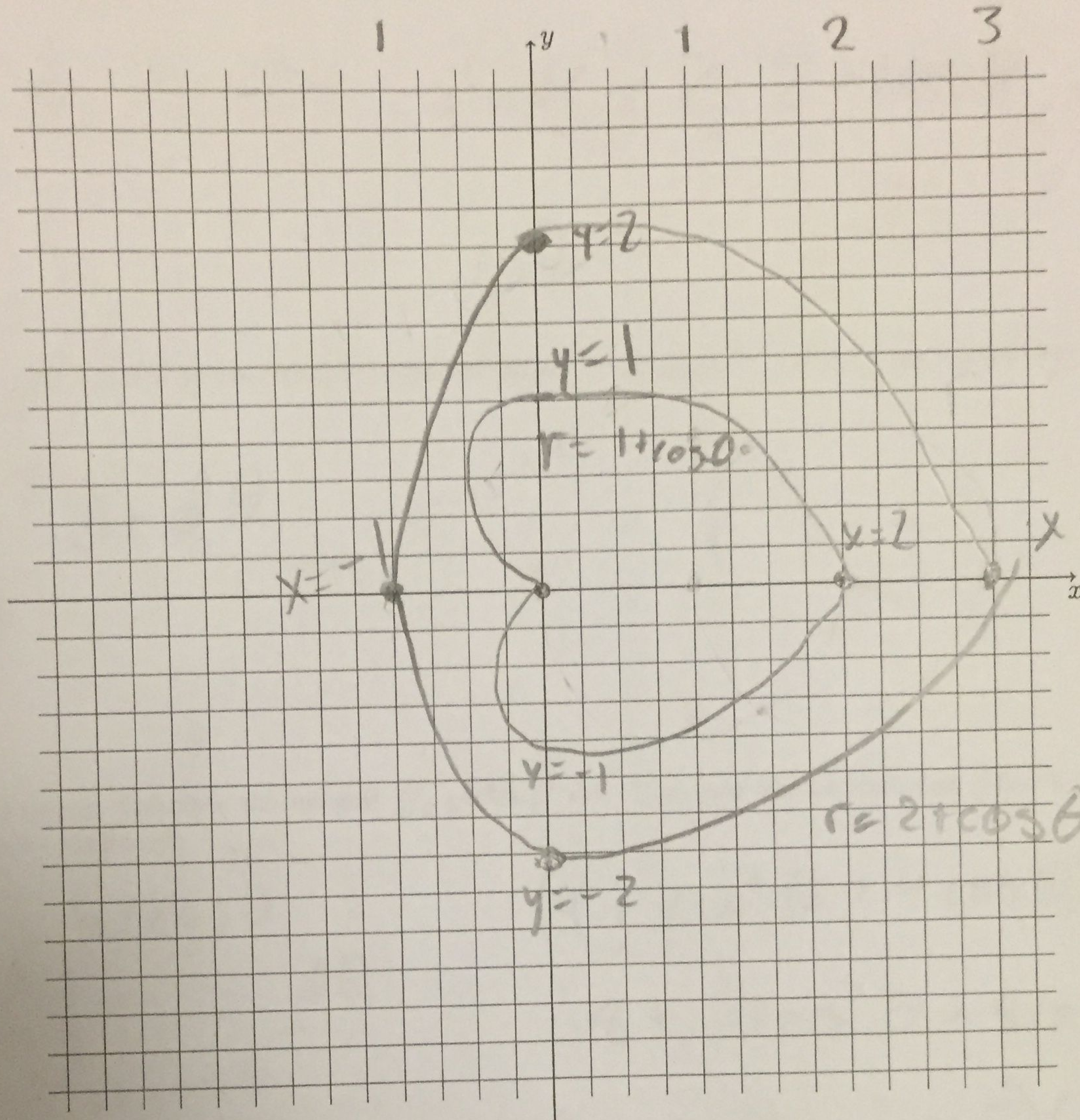
3. In this question, consider the curves $r = 1 + \cos\theta$ and $r = 2 + \cos\theta$.

   (a) (4 points) Sketch both the curves on the graph provided. Make sure to indicate where the curves cross the axes.

$\frac{\sqrt{2}}{2} + 1 \approx$

1.7

4



(b) (2 points) Write the region *between* the two curves as a radially simple region, i.e in the form $\varphi \leq \theta \leq \psi$ and $r_1(\theta) \leq r \leq r_2(\theta)$ for some functions r_1 and r_2.

$0 \leq \theta \leq 2\pi \quad 1+\cos\theta \leq r \leq 2+\cos\theta$

2

(c) (2 points) Let D be the region between the curves. Write $\iint_D \sqrt{x^2+y^2}\, dA$ as an iterated integral.

$$\int_0^{2\pi}\int_{1+\cos\theta}^{2+\cos\theta} r^2 \, dr\, d\theta$$

✓

(d) (4 points) Calculate the integral $\iint_D \sqrt{x^2+y^2}\, dA$. You may use the fact that $\int \cos^2\theta\, d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$.

$$\int_0^{2\pi} \left[\frac{1}{3} r^3\right]_{1+\cos\theta}^{2+\cos\theta} d\theta$$

$(4+\cos^2\theta+4\cos\theta)(2+\cos\theta)$

$8+6\cos^2\theta+8\cos\theta+\cos^3\theta$

$-(1+\cos^2\theta+2\cos\theta)(1+\cos\theta)$

$1+3\cos^2\theta+2\cos\theta+\cos^3\theta$

$$= \int_0^{2\pi} \frac{7}{3} + 1\cos^2\theta + 2\cos\theta \, d\theta$$

$$= \left[\frac{7}{3}\theta + \frac{1}{2}(\theta + \sin\theta\cos\theta) + 2\sin\theta\right]_0^{2\pi}$$

$$= \frac{14\pi}{3} + \pi = \boxed{\frac{17\pi}{3}}$$

✓

4. Consider the region $\mathcal{E}$ in the intersection of the two balls $x^2+y^2+(z-\frac{1}{2})^2 \le 1$ and $x^2+y^2+(z+\frac{1}{2})^2 \le 1$.

(a) (4 points) Describe the region in the form

$$\mathcal{E} = \{\, (x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \mathcal{D},\ z_1(x,y) \le z \le z_2(x,y)\}$$

for $\mathcal{D}$ a region in the xy-plane. Your answer should specify what $\mathcal{D}$ is.

$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2 = x^2+y^2+\left(z+\tfrac{1}{2}\right)^2$$

$$z^2 - z + \tfrac{1}{4} = z^2 + z + \tfrac{1}{4}$$

$$-z = z$$

$$z = 0$$

$$x^2+y^2+\left(\tfrac{1}{2}\right)^2 \le 1$$

$$x^2+y^2 \le \tfrac{3}{4}$$

$$\mathcal{E} = \left\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in x^2+y^2 \le \tfrac{3}{4},\ x^2+y^2+\left(z+\tfrac{1}{2}\right)^2 \le z \le x^2+y^2+\left(z-\tfrac{1}{2}\right)^2\right\}$$

4

(b) (5 points) Compute the volume of the region $\mathcal{E}$.

$$\int_{x=-\sqrt{\frac{3}{4}}}^{\sqrt{\frac{3}{4}}} \int_{y=-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}} \int_{z=\sqrt{1-x^2-y^2}+\frac{1}{2}}^{z=\sqrt{1-x^2-y^2}-\frac{1}{2}} dz\,dy\,dx$$

4

$$\left(z+\tfrac{1}{2}\right)^2 = 1-x^2-y^2$$

$$z = \sqrt{1-x^2-y^2} - \tfrac{1}{2}$$

$$\left(z-\tfrac{1}{2}\right)^2 = 1-x^2-y^2$$

$$z = \sqrt{1-x^2-y^2} + \tfrac{1}{2}$$

$$= \int_{-\sqrt{\frac{3}{4}}}^{\sqrt{\frac{3}{4}}} \int_{-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}} \frac{1}{2} - -\frac{1}{2}\, dy\,dz$$

$$\int_{-\sqrt{\frac{3}{4}}}^{\sqrt{\frac{3}{4}}} [y]_{-\sqrt{\frac{3}{4}-y^2}}^{\sqrt{\frac{3}{4}-x^2}}\, dx = \int_{-\sqrt{\frac{3}{4}}}^{\sqrt{\frac{3}{4}}} 2\sqrt{\tfrac{3}{4}-x^2}\, dx$$

More on next page

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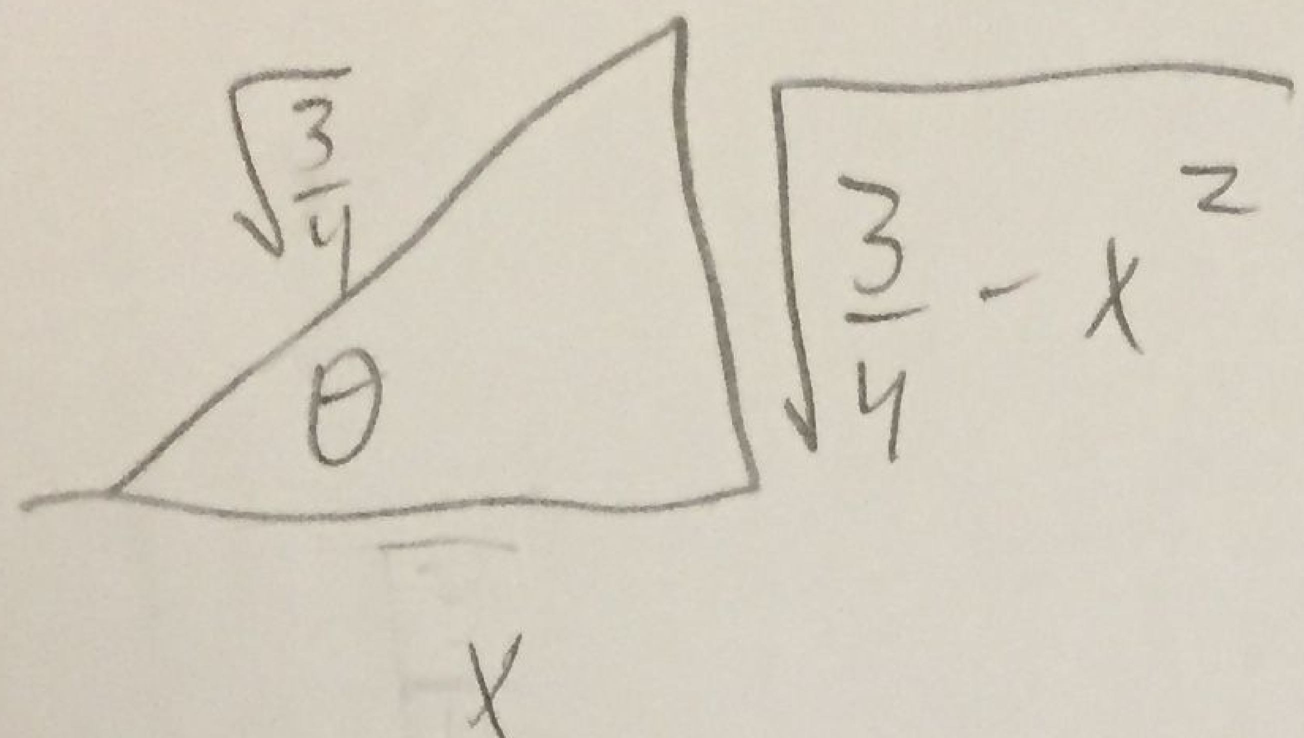

$$\cos\theta = \frac{x}{\sqrt{\frac{3}{4}}}$$

$$x = \sqrt{\frac{3}{4}}\cos\theta$$

$$\frac{dx}{d\theta} = -\sqrt{\frac{3}{4}}\sin\theta$$

$$dx = -\sqrt{\frac{3}{4}}\sin\theta\, d\theta$$

$$\sin\theta = \frac{\sqrt{\frac{3}{4}-x^2}}{\sqrt{\frac{3}{4}}}$$

$$\int -2\sqrt{\frac{3}{4}}\sin\theta\sqrt{\frac{3}{4}}\sin\theta\, d\theta$$

$$= \int -\frac{3}{2}\sin^2\theta\, d\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$x = \sqrt{\frac{3}{4}} \to \cos\theta = 1$$
$$\theta = 0$$
$$x = -\sqrt{\frac{3}{4}} \to \cos\theta = -1$$
$$\theta = \pi$$

$$= \int -\frac{3}{4}(1-\cos 2\theta)\, d\theta$$

$$= -\frac{3}{4}\theta + \frac{3}{8}\sin 2\theta \Big|_{\theta=0}^{\theta=\pi} = \boxed{\frac{3}{4}\pi}$$