Math 32B Final

Professor: Noah White / Term: Spring 2020 / Score: 62/62

Problem 1a *(2 points)*

Consider the region in $\mathbb{R}^3$ bounded by the planes $x + y + z = 1$ and $2x + 2y + z = 2$ in the first octant (i.e. where $x, y, z \geq 0$).

Let $\mathcal{S}$ be the surface that is the boundary of this region, with outward pointing normal vectors.

Calculate the divergence of the vector field $\mathbf{F} = \langle xz, 3yz, 2z^2 \rangle$.

We know that the *divergence* of a vector field is given by $\nabla \cdot \mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$. In our case,

$$\nabla \cdot \mathbf{F} = z + 3z + 4z = \boxed{8z}$$

Problem 1b *(8 points)*

Calculate the flux of $\mathbf{F}$ through the surface $\mathcal{S}$.

We know the flux of $\mathbf{F}$ through $\mathcal{S}$ is the *vector surface integral* of $\mathbf{F}$ over $\mathcal{S}$. However, calculating the vector surface integral would be difficult in this case, as we would have to parametrize the four distinct surfaces that comprise $\mathcal{S}$.

Instead, we can try using the *divergence theorem*, which states that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \nabla \cdot \mathbf{F} \, dV$$

$\mathcal{S}$ is oriented *outwards*, so we don't have to adjust the sign of the theorem.

Notice that we can write the two planes as functions of z, namely $z_1(x, y) = 1 - x - y$ and $z_2(x, y) = 2 - 2x - 2y$. This means that $\mathcal{W}$ is one contiguous z-*simple* region, meaning it will be much easier to find the triple integral over $\mathcal{W}$ than to find the surface integral over its boundary $\mathcal{S}$. It also helps that $\nabla \cdot \mathbf{F}$ is really easy to integrate, since it is just $8z$.

Both planes intersect each other when $1 - x - y$ = $2 - 2x - 2y$, or when $y = 1 - x$.

So $\mathcal{W}$ projects onto a triangle $\mathcal{D}$ in the xy-plane defined by $0 \leq x \leq 1, \; 0 \leq y \leq 1 - x$ (which is entirely in the first quadrant).

Our *z*-bounds are simply $z_1(x, y) \leq z \leq z_2(x, y) \implies 1 - x - y \leq z \leq 2 - 2x - 2y$.

Now that we have found the bounds for $\mathcal{W}$, we can rewrite the triple integral as an iterated integral:

$$\iiint_{\mathcal{W}} \nabla \cdot \mathbf{F} \, dV = \int_0^1 \int_0^{1-x} \int_{1-x-y}^{2-2x-2y} 8z \, dz \, dy \, dx$$

All that is left to do is evaluate the integral, from the inside out. First evaluate the z-integral:

$$\int_0^1 \int_0^{1-x} \int_{1-x-y}^{2-2x-2y} 8z \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} \left[4z^2\right]_{1-x-y}^{2-2x-2y} dy \, dx = 4 \int_0^1 \int_0^{1-x} (2(1-x-y))^2 - (1-x-y)^2 \, dy \, dx$$

For the y-integral, we can make the substitution $u = 1 - x - y, du = -dy$ to greatly simplify the integral.

$$4 \int_0^1 \int_0^{1-x} 3(1-x-y)^2 \, dy \, dx = -4 \int_0^1 \int_{1-x}^0 3u^2 \, du \, dx = -4 \int_0^1 \left[u^3\right]_{1-x}^0 dx = -4 \int_0^1 -(1-x)^3 \, dx$$

We can do the same for the x-integral, by making the substitution $u = 1 - x, du = -dx$.

$$-4 \int_1^0 u^3 du = -4 \left[\frac{u^4}{4}\right]_1^0 = -4 \left[-\frac{1}{4}\right] = \boxed{1}$$

Problem 2a *(3 points)*

Consider the region $\mathcal{P}$ in $\mathbb{R}^2$ bounded by the lines $y = -x, y = 3 - x, y = 2x$ and $y = 2x - 6$. This is a parallelogram.

Find a linear change of coordinates $G(u, v)$ such that G maps the rectangle $[0, 3] \times [0, 6]$ to $\mathcal{P}$. Calculate the Jacobian of G.

We can rewrite the bounds of $\mathcal{P}$ as $x + y = 0, x + y = 3, 2x - y = 0$ and $2x - y = 6$.

So the region $\mathcal{P}$ we care about is defined by $\begin{cases} 0 \leq x + y \leq 3 \\ 0 \leq 2x - y \leq 6 \end{cases}$

If we let $\begin{cases} u = x + y \\ v = 2x - y \end{cases}$, then the corresponding region in the uv-plane is the rectangle $[0, 3] \times [0, 6]$.

This change of coordinates is actually $G^{-1}(x, y)$, since we are mapping the (x, y) coordinates to (u, v) coordinates. We are trying to find the reverse, $G(u, v)$, which maps (u, v) coordinates to (x, y) coordinates.

To do this, we must solve for (x, y). From the top equation, we can see that $x = u - y$. Plugging the expression for x into the bottom equation, we get $v = 2(u - y) - y \implies v = 2u - 2y - y \implies -3y = -2u + v \implies y = \frac{2}{3}u - \frac{1}{3}v$.

Now we need to plug in for y in the top equation, to get $x = u - (\frac{2}{3}u - \frac{1}{3}v) \implies x = \frac{1}{3}u + \frac{1}{3}v$.

So our linear change of coordinates would be

$$\boxed{G(u, v) = (\frac{1}{3}u + \frac{1}{3}v, \frac{2}{3}u - \frac{1}{3}v)}$$

We know that the *Jacobian* of a map $G(u, v)$ is

$$\text{Jac}(G) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

In our case,

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(\frac{1}{3}\right)\left(-\frac{1}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = -\frac{1}{9} - \frac{2}{9} = -\frac{3}{9} = \boxed{-\frac{1}{3}}$$

Problem 2b *(3 points)*

Calculate the integral $\iint_{\mathcal{P}} x^2 - y^2\ dA$.

When we change variables, we must adjust the integral to be in terms of the new variables. We know that

$$\iint_{\mathcal{P}} f(x,y)\ dA_{xy} = \iint_{\mathcal{R}} f(x(u,v), y(u,v))\ |\text{Jac}(G)|\ dA_{uv}$$

From part *(a)*, $f(x(u,v), y(u,v)) = f(\frac{1}{3}u + \frac{1}{3}v, \frac{2}{3}u - \frac{1}{3}v) = (\frac{1}{3}u + \frac{1}{3}v)^2 - (\frac{2}{3}u - \frac{1}{3}v)^2$, and $|\text{Jac}(G)| = \frac{1}{3}$.

So we have

$$\frac{1}{3}\iint_{\mathcal{R}} (\frac{1}{3}u + \frac{1}{3}v)^2 - (\frac{2}{3}u - \frac{1}{3}v)^2\ dA_{uv} = \frac{1}{27}\int_0^3 \int_0^6 (u+v)^2 - (2u-v)^2\ dv\ du$$
$$= \frac{1}{27}\int_0^3 \int_0^6 u^2 + 2uv + v^2 - (4u^2 - 4uv + v^2)\ dv\ du = \frac{1}{27}\int_0^3 \int_0^6 -3u^2 + 6uv\ dv\ du$$
$$= \frac{1}{27}\int_0^3 \Big[-3u^2 v + 3uv^2 \Big]_0^6 du = \frac{1}{27}\int_0^3 -18u^2 + 108u\ du = \frac{1}{27}\Big[-6u^3 + 54u^2 \Big]_0^3$$
$$= \frac{1}{27}\Big[-6(3)^3 + 54(3)^2 \Big] = -6 + 18 = \boxed{12}$$

Problem 2c *(4 points)*

Now consider the region $\mathcal{E} \subset \mathbb{R}^3$ bounded by the planes $y = -x, y = 3 - x, y = 2x$ and $y = 2x - 6$, and by the xy-plane below and the surface $z = 9 + x^2 - y^2$ above. Calculate the volume of $\mathcal{E}$.

We know that the volume of a region $\mathcal{E}$ is the triple integral of the constant function $f(x, y, z) = 1$ over $\mathcal{E}$.

We can see that $\mathcal{E}$ is a z-simple region between $z_1(x, y) = 0$ and $z_2(x, y) = 9 + x^2 - y^2$. Since the four planes that are the bounds of $\mathcal{E}$ are also the bounds of $\mathcal{P}$ from part *(b)*, we can see that $\mathcal{P}$ is the projection of $\mathcal{E}$ onto the xy-plane.

So we have:

$$\text{Volume}(\mathcal{E}) = \iiint_{\mathcal{E}} 1 \, dV = \iint_{\mathcal{P}} \int_0^{9+x^2-y^2} 1 \, dz \, dA_{xy} = \iint_{\mathcal{P}} 9 + x^2 - y^2 \, dA_{xy} = \iint_{\mathcal{P}} 9 \, dA_{xy} + \iint_{\mathcal{P}} x^2 - y^2 \, dA_{xy}$$

We already know what $\iint_{\mathcal{P}} x^2 - y^2 \, dA_{xy}$ is from part *(b)*, we just need to find $\iint_{\mathcal{P}} 9 \, dA_{xy}$.

Using the same change of variables formula as before, we get

$$\iint_{\mathcal{P}} 9 \, dA_{xy} = \iint_{\mathcal{R}} 9 \left(\frac{1}{3}\right) dA_{uv} = 3 \iint_{\mathcal{R}} 1 \, dA_{uv}$$

We know that the double integral of the constant function $f(x, y) = 1$ over $\mathcal{R}$ is just the area of $\mathcal{R}$. $\mathcal{R}$ is just a rectangle with width 3 and height 6, meaning its area is 18. So

$$\iint_{\mathcal{P}} 9 \, dA_{xy} = 3 \cdot area(\mathcal{R}) = 3 \cdot 18 = 54$$

Finally, combine the two integrals together and we're done:

$$\text{Volume}(\mathcal{E}) = \iint_{\mathcal{P}} 9 \, dA_{xy} + \iint_{\mathcal{P}} x^2 - y^2 \, dA_{xy} = 54 + 12 = \boxed{66}$$

Problem 3a *(5 points)*

Consider the region $\mathcal{R}$ of the plane described by $(\sqrt[3]{x})^2 + (\sqrt[3]{(y/8)})^2 \leq 1$. Use Green's Theorem to calculate the area of $\mathcal{R}$.

Green's Theorem states that $\iint_{\mathcal{D}} \operatorname{curl}(\mathbf{F})\, dA = \oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r}$.

The area of $\mathcal{R}$ is $\iint_{\mathcal{R}} 1\, dA$. If we find $\mathbf{F}$ such that $\operatorname{curl}(\mathbf{F}) = 1$, then by Green's Theorem, $\operatorname{area}(\mathcal{R}) = \oint_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{r}$. One such field is $\mathbf{F} = \langle 0, x \rangle$.

We can parameterize $\partial\mathcal{R}$ with the parameterization $\mathbf{r}(t) = (\cos^3(t), 8\sin^3(t))$, where $t \in [0, 2\pi]$.

We know that $\oint_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{r}$ can be rewritten as $\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$.

$\mathbf{F}(\mathbf{r}(t)) = \langle -y, 0 \rangle = \langle -8\sin^3(t), 0 \rangle$ and $\mathbf{r}'(t) = (-3\sin(t)\cos^2(t), 24\cos(t)\sin^2(t))$, so $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 24\sin^4(t)\cos^2(t)$.

We can use linearity and trigonometric identities to separate the integral into two simpler ones:

$$\int_0^{2\pi} 24\sin^4(t)\cos^2(t)\, dt = 24\int_0^{2\pi} \sin^4(t)(1-\sin^2(t))\, dt = 24\int_0^{2\pi} \sin^4(t) - \sin^6(t)\, dt = 24\left[\int_0^{2\pi} \sin^4(t)\, dt - \int_0^{2\pi} \sin^6(t)\, dt\right]$$

We can solve $\int_0^{2\pi} \sin^4(t)\, dt$ first using the sine reduction formula:

$$\int \sin^n(t)\, dt = -\frac{1}{n}\sin^{n-1}(t)\cos(t) + \frac{n-1}{n}\int \sin^{n-2}(t)\, dt$$

So we have:

$$\int_0^{2\pi} \sin^4(t)\, dt = \left[-\frac{1}{4}\sin^3(t)\cos(t) + \frac{3}{4}\int \sin^2(t)\, dt\right]_0^{2\pi}$$

We can use the sine reduction formula again to solve for $\frac{3}{4}\int \sin^2(t)\, dt$:

$$\int \sin^2(t)\, dt = -\frac{1}{2}\sin(t)\cos(t) + \frac{1}{2}\int 1\, dt = \frac{1}{2}(t - \sin(t)\cos(t))$$

Now plug the result back into the original integral and evaluate, taking advantage of the fact that $\sin(t)$ and $\sin^3(t)$ are odd:

$$\left[-\frac{1}{4}\sin^3(t)\cos(t) + \frac{3}{4}\int \sin^2(t)\, dt\right]_0^{2\pi} = \left[-\frac{1}{4}\sin^3(t)\cos(t) + \frac{3}{8}(t - \sin(t)\cos(t))\, dt\right]_0^{2\pi} = \frac{3}{8}(2\pi) = \frac{3\pi}{4}$$

Now let's solve for $\int_0^{2\pi} \sin^6(t)\, dt$, again using the sine reduction formula and using odd functions to our advantage:

$$\int_0^{2\pi} \sin^6(t)\, dt = \left[-\frac{1}{6}\sin^5(t)\cos(t) + \frac{5}{6}\int \sin^4(t)\, dt\right]_0^{2\pi}$$

$$= \left[-\frac{1}{6}\sin^5(t)\cos(t) + \frac{5}{6}\left(-\frac{1}{4}\sin^3(t)\cos(t) + \frac{3}{8}(t - \sin(t)\cos(t))\right)\right]_0^{2\pi}$$

$$= \left[-\frac{1}{6}\sin^5(t)\cos(t) - \frac{5}{24}\sin^3(t)\cos(t) - \frac{15}{48}\sin(t)\cos(t) + \frac{15}{48}t\right]_0^{2\pi} = \frac{15}{48}(2\pi) = \frac{15\pi}{24}$$

Finally, add the two integrals together and we're done:

$$= 24\left[\int_0^{2\pi} \sin^4(t)\, dt - \int_0^{2\pi} \sin^6(t)\, dt\right] = 24(\frac{3\pi}{4} - \frac{15\pi}{24}) = 18\pi - 15\pi = \boxed{3\pi}$$

Problem 3b *(5 points)*

A spiral ramp is 10 feet wide and in one complete 2π revolution it goes up π feet. What is the area of this spiral ramp?

The surface area of our surface is given by the surface integral $\iint_{\mathcal{D}} \|\mathbf{N}(u,v)\|\, dA_{uv}$.

We can parameterize this surface in two variables, u and v. We want u to parameterize points on the line segment at a height v, and we want v to parameterize points on the helix at a point u on the line segment.

One such parameterization is $G(u,v) = (u\cos(v), u\sin(v), \frac{v}{2})$, where $u \in [0,10]$ and $v \in [0, 2\pi]$.

We must now find the normal vector $\mathbf{N}(u,v)$, which is the cross product of the two tangent vectors, $\mathbf{T}_u(G(u,v)) \times \mathbf{T}_v(G(u,v))$.

The tangent vectors are:

$$\mathbf{T}_u(G(u,v)) = \frac{\partial}{\partial u}(u\cos(v), u\sin(v), \frac{v}{2}) = \langle \cos(v), \sin(v), 0\rangle$$
$$\mathbf{T}_v(G(u,v)) = \frac{\partial}{\partial v}(u\cos(v), u\sin(v), \frac{v}{2}) = \langle -u\sin(v), u\cos(v), \frac{1}{2}\rangle$$

Computing the cross product, we get our normal vector:

$$\langle \cos(v), \sin(v), 0\rangle \times \langle -u\sin(v), u\cos(v), \frac{1}{2}\rangle = \langle \frac{1}{2}\sin(v), -\frac{1}{2}\cos(v), u\rangle$$

The norm of the normal vector is $\|\mathbf{N}(u,v)\| = \sqrt{(\frac{1}{2}\sin(v))^2 + (-\frac{1}{2}\cos(v))^2 + u^2} = \sqrt{u^2 + \frac{1}{4}}$.

Plugging in all the necessary info into our surface integral, we can simplify it into an iterated integral:

$$\iint_{\mathcal{D}} \|\mathbf{N}(u,v)\|\, dA_{uv} = \int_0^{2\pi}\int_0^{10} \sqrt{u^2 + \frac{1}{4}}\, du\, dv$$

We can use the hint from HW 3, which states that $\frac{d}{dt}(t\sqrt{t^2+1} + \sinh^{-1}(t)) = 2\sqrt{t^2+1}$. To take advantage of this, we must make a substitution, namely $w = 2u, dw = 2\,du$, to fully evaluate the integral:

$$\int_0^{2\pi}\int_0^{10} \sqrt{u^2+\frac{1}{4}}\, du\, dv = \frac{1}{2}\int_0^{2\pi}\int_0^{20}\sqrt{\left(\frac{w}{2}\right)^2 + \frac{1}{4}}\, dw\, dv = \frac{1}{2}\int_0^{2\pi}\int_0^{20}\sqrt{\frac{w^2}{4}+\frac{1}{4}}\, dw\, dv = \frac{1}{8}\int_0^{2\pi}\int_0^{20}\sqrt{w^2+1}\, dw\, dv$$
$$= \frac{1}{8}\int_0^{2\pi}\left[w\sqrt{w^2+1} + \sinh^{-1}(w)\right]_0^{20} dv = \frac{1}{8}\int_0^{2\pi} 20\sqrt{401} + \sinh^{-1}(20)\, dv = \boxed{\frac{\pi}{4}(20\sqrt{401} + \sinh^{-1}(20))}$$

Problem 4a *(2 points)*

Consider the vector field $\mathbf{F} = \langle xy, yz, zx \rangle$. Calculate the curl of $\mathbf{F}$.

We know that the *curl* of a vector field is given by
$\nabla \times \mathbf{F} = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$. In our case,

$$\nabla \times \mathbf{F} = \boxed{\langle -y, -z, -x \rangle}$$

Problem 4b *(8 points)*

Consider the closed curve $\mathcal{C}$ determined by the intersction of the cylinder $x^2 + y^2 = 1$ and the surface $z = x^2$, with orientation counter-clockwise when looking from above. Calculate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

We can't use the fundamental theorem of conservative vector fields in this case, since $\nabla \times \mathbf{F} \neq 0$. We can still try parameterizing the line and see if it's possible to integrate it.

From the equation of the cylinder, we can parameterize x and y pretty easily, using $x(t) = \cos(t)$ and $y(t) = \sin(t)$. z is a function of x, so a parameterization of z would be $z(t) = \cos^2(t)$. So our parameterization looks like $\mathbf{r}(t) = (\cos(t), \sin(t), \cos^2(t))$, where $t \in [0, 2\pi]$. So the line integral can be expressed as $\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$. Let's find $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$ first:

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin(t), -\cos^2(t), -\cos(t)\rangle$$
$$\mathbf{r}'(t) = (-\sin(t), \cos(t), -2\sin(t)\cos(t))$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \sin^2(t) - \cos^2(t) + 2\sin(t)\cos^2(t)$$

Now we can evaluate the integral by breaking it up into simpler integrals:

$$\int_0^{2\pi} \sin^2(t) - \cos^2(t) + 2\sin(t)\cos^2(t)\, dt = \int_0^{2\pi} 1 - 2\sin^2(t) + 2\sin(t)\cos^2(t)\, dt$$
$$= \int_0^{2\pi} 1\, dt - 2\int_0^{2\pi} \sin^2(t)\, dt + 2\int_0^{2\pi} \sin(t)\cos^2(t)\, dt$$

Since $\sin(t)$ is an odd function, integrating $\sin(t)\cos^2(t)$ over $[0, 2\pi]$ gives us zero, so we can remove that term entirely. Finally, we can finish evaluating our integrals:

$$\int_0^{2\pi} 1\, dt - 2\int_0^{2\pi} \sin^2(t)\, dt = 2\pi - 2\left[\frac{1}{2}(t - \sin(t)\cos(t))\right]_0^{2\pi} = 2\pi - 2\pi = \boxed{0}$$

Problem 5a *(2 points)*

Consider the vector field: $\mathbf{F}(x,y,z) = \langle \frac{e^{z^2}}{1+y^2+z^2}, \frac{1}{1+x^4}, z^2+1 \rangle$. What is the divergence of $\mathbf{F}$?

The divergence of the vector field is

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 2z = \boxed{2z}$$

Problem 5b *(8 points)*

Let $\mathcal{S}$ be the hemisphere $x^2 + y^2 + z^2 = 1$ where $z \geq 0$ with outward pointing orientation. What is $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$?

Since $\mathbf{F}$ looks very complex, finding the surface integral directly might be extremely difficult in this case. However, we can take advantage of the Divergence Theorem to relate $\mathcal{S}$ with the volume it encloses above the xy-plane. We can write:

$$\iiint_{\mathcal{W}} \nabla \cdot \mathbf{F}\, dV = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{S} \implies \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \nabla \cdot \mathbf{F}\, dV - \iint_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{S}$$

We have to add an additional surface $\mathcal{D}$, which is the domain contained within the unit circle, in order to fully enclose the region $\mathcal{W}$, since the Divergence Theorem only works on closed regions. Thankfully, taking the surface integral over $\mathcal{D}$ will be a lot easier since it lies on the xy-plane, meaning any expression involving z will be a lot simpler.

First, let's calculate $\iiint_{\mathcal{W}} \nabla \cdot \mathbf{F}\, dV$. We can use spherical coordinates to integrate over $\mathcal{W}$:

$$\iiint_{\mathcal{W}} 2z\, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 2\rho \cos(\phi)\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 2\rho^3 \cos(\phi) \sin(\phi)\, d\rho\, d\phi\, d\theta$$

We need to make a u-substitution ($u = \sin(\phi), du = \cos(\phi)\, d\phi$) to get our answer:

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos(\phi) \sin(\phi) \left[\frac{\rho^4}{2}\right]_0^1 d\phi\, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos(\phi) \sin(\phi)\, d\phi\, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^1 u\, du\, d\theta$$
$$= \frac{1}{2} \int_0^{2\pi} \left[\frac{u^2}{2}\right]_0^1 d\theta = \frac{1}{4} \int_0^{2\pi} 1\, d\theta = \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

Now we need to find $\iint_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{S}$ by finding a surface parameterization of $\mathcal{D}$, as well as the normal vector to $\mathcal{D}$.

Finding a surface parameterization is pretty simple since it's just the region enclosed within the unit circle in the xy-plane; we can just use polar coordinates. The parameterization we'll use is $G(r, \theta) = (r\cos(\theta), r\sin(\theta), 0)$.

Let's find the normal vector by crossing the two tangent vectors:

$$\mathbf{T}_r(G(r,\theta)) = \frac{\partial}{\partial r}(r\cos(\theta), r\sin(\theta), 0) = \langle \cos(\theta), \sin(\theta), 0 \rangle$$
$$\mathbf{T}_\theta(G(r,\theta)) = \frac{\partial}{\partial \theta}(r\cos(\theta), r\sin(\theta), 0) = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle$$
$$\mathbf{N}(r,\theta) = \mathbf{T}_r \times \mathbf{T}_\theta = \langle \cos(\theta), \sin(\theta), 0 \rangle \times \langle -r\sin(\theta), r\cos(\theta), 0 \rangle = \langle 0, 0, r \rangle$$

We're not done yet! The normal vector is pointing in the *wrong direction*, since it should be oriented pointing *outwards* from the surface, not *into* the surface! To get the correct orientation, we need to take the cross product in reverse order:

$$\mathbf{N}(r,\theta) = \mathbf{T}_\theta \times \mathbf{T}_r = \langle -r\sin(\theta), r\cos(\theta), 0\rangle \times \langle \cos(\theta), \sin(\theta), 0\rangle = \langle 0, 0, -r\rangle$$

Now we can finally take the surface integral:

$$\iint_{\mathcal{D}} \mathbf{F}(G(r,\theta)) \cdot \mathbf{N}(r,\theta)\; du\; dv = \int_0^{2\pi}\int_0^1 \langle 0, \frac{1}{1+r^4\cos^4(\theta)}, 1\rangle \cdot \langle 0,0,-r\rangle\; dr\; d\theta = \int_0^{2\pi}\int_0^1 -r\; dr\; d\theta$$
$$= \int_0^{2\pi}\left[\frac{-r^2}{2}\right]_0^1 d\theta = -\frac{1}{2}\int_0^{2\pi} 1\; d\theta = -\frac{1}{2}(2\pi) = -\pi$$

Plugging in both integrals into the original expression, we get our final answer:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \nabla \cdot \mathbf{F}\; dV - \iint_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \boxed{\frac{3\pi}{2}}$$

Problem 6a *(2 points)*

Consider the vector field

$$\mathbf{F}(x,y,z) = \langle x, \frac{-z}{y^2+z^2}, \frac{y}{y^2+z^2} \rangle$$

with domain $\mathbb{R}^3 \setminus \{(x,0,0) \mid x \in \mathbb{R}\}$. Calculate the curl of $\mathbf{F}$.

The curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle = \langle \frac{z^2-y^2}{(y^2+z^2)^2} - \frac{z^2-y^2}{(y^2+z^2)^2}, 0-0, 0-0 \rangle = \boxed{\mathbf{0}}$$

Problem 6b *(3 points)*

Is $\mathbf{F}$ conservative? Demonstrate your answer with a calculation.

Although $\nabla \times \mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is not simply connected, so we can't make a conclusion based on that. We can instead try to find a closed curve $\mathcal{C}$ such that $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \neq 0$. This would show that $\mathbf{F}$ is *not* conservative, since the vector line integral of any closed loop on a conservative vector field must be zero.

Let's try the unit circle in the yz-plane. We can parameterize this pretty easily, using $\mathbf{r}(t) = (0, \cos(t), \sin(t))$, where $t \in [0, 2\pi]$. With this parameterization, let's calculate the vector line integral:

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} \langle 0, -\sin(t), \cos(t)\rangle \cdot \langle 0, -\sin(t), \cos(t)\rangle\, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = \int_0^{2\pi} 1\, dt = \boxed{2\pi}$$

Since the vector line integral is nonzero, the vector field $\mathbf{F}$ is therefore **not conservative**.

Problem 6c *(7 points)*

Consider the oriented curve $\mathcal{C}$ given by the parameterization $\mathbf{r}(t) = (t, \cos(2\pi t), 2\sin(2\pi t))$, where $t \in [0, 2]$.

Calculate the integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Our curve $\mathcal{C}$ traces out a portion of an *elliptical spiral* around the x-axis. It makes two full revolutions around the x-axis, and its ending position is 2 units in the positive x-direction past its starting point, while its y-coordinate and z-coordinate are the same. While we can take the vector line integral directly, the elliptical nature of the parameterization makes this a bit awkward. Instead, we can try using curve arithmetic to our advantage, and use the fact that $\mathbf{F}$ is conservative on *simply connected* domains, to find an easier curve to integrate that has the same vector line integral.

In the pictures below, I drew our curve $\mathcal{C}$, shown in red, and an additional curve, $\mathcal{S}$, shown in blue, parameterized by the curve $\mathbf{r}(t) = (t, \cos(2\pi t), \sin(2\pi t))$, where $t \in [0, 2]$.

Let's divide $\mathcal{C}$ into four smaller curves, each defined over a specific t-domain. Let $\mathcal{C}_1$ be the part of $\mathcal{C}$ where $t \in [0, 0.5]$, let $\mathcal{C}_2$ be the part of $\mathcal{C}$ where $t \in [0.5, 1]$, let $\mathcal{C}_3$ be the part of $\mathcal{C}$ where $t \in [1, 1.5]$, and let $\mathcal{C}_4$ be the part of $\mathcal{C}$ where $t \in [1.5, 2]$.
$\mathcal{S}$ will also be divided into four analogous curves, labeled $\mathcal{S}_1$ through $\mathcal{S}_4$, along the same t-boundaries.

Each $\mathcal{C}_n - \mathcal{S}_n$ pair forms its own closed loop, which we'll call $\mathcal{L}_n$. $\mathcal{L}_n$ is defined by traveling in the positive t-direction along $\mathcal{C}_n$ and the *negative* t-direction along $\mathcal{S}_n$. In the picture below, we can see $\mathcal{L}_1$ and $\mathcal{L}_2$.

$\mathcal{L}_1$ (shown in orange), along with $\mathcal{L}_3$, are both contained in the domain $\{\mathbb{R}^3 \mid z \geq 0\}$, minus the x-axis.
$\mathcal{L}_2$ (shown in green), along with $\mathcal{L}_4$, are both contained in the domain $\{\mathbb{R}^3 \mid z \leq 0\}$, minus the x-axis.

Since both domains are simply connected, $\mathbf{F}$ is conservative on both domains, so $\displaystyle\int_{\mathcal{L}_n} \mathbf{F} \cdot d\mathbf{r} = 0$.

By curve arithmetic, $\mathcal{L}_n = \mathcal{C}_n - \mathcal{S}_n$. So we can expand the integral to show that the vector line integrals of $\mathcal{C}_n$ and $\mathcal{S}_n$ are equal:

$$\int_{\mathcal{L}_n} \mathbf{F} \cdot d\mathbf{r} = 0 \implies \int_{\mathcal{C}_n - \mathcal{S}_n} \mathbf{F} \cdot d\mathbf{r} = 0 \implies \int_{\mathcal{C}_n} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathcal{S}_n} \mathbf{F} \cdot d\mathbf{r} = 0 \implies \int_{\mathcal{C}_n} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}_n} \mathbf{F} \cdot d\mathbf{r}$$

We can then go on to show that the vector line integrals of $\mathcal{C}$ and $\mathcal{S}$ are equal:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r}$$
$$= \int_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{S}_3} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{S}_4} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{r}$$

Now all we have to do is find $\int_S \mathbf{F} \cdot d\mathbf{r}$. Our parameterization from before is $\mathbf{r}(t) = (t, \cos(2\pi t), \sin(2\pi t))$, where $t \in [0, 2]$.

$\mathbf{F}(\mathbf{r}(t)) = \langle t, -\sin(2\pi t), \cos(2\pi t)\rangle$, and $\mathbf{r}'(t) = (1, -2\pi sin(2\pi t), -2\pi cos(2\pi t))$. The dot product is:

$$\mathbf{F}(\mathbf{r}(t)) = \langle t, -\sin(2\pi t), \cos(2\pi t)\rangle \cdot \langle 1, -2\pi \sin(2\pi t), -2\pi \cos(2\pi t)\rangle = t + 2\pi$$

Now we can evaluate the integral to get our final answer:

$$\int_0^2 t + 2\pi \, dt = \left[\frac{t^2}{2} + 2\pi t\right]_0^2 = \frac{2^2}{2} + 2\pi(2) = \boxed{2 + 4\pi}$$