UCLA: Math 32B **Final Exam** **Spring, 2020**

Upload your solutions to gradescope for the following questions by 11:59pm LA time on Tuesday June 9.

- Late exams will not be accepted.
- Your scans must be readable and good quality. Use good lighting and a scanning app.
- Questions 1,2,3 must begin on a new page and questions must be allocated correctly on Gradescope.
- Write your solutions **linearly**. We should be able to easily read your solutions and do not want to hunt around the page for it.

1. Consider the region in $\mathbb{R}^3$ bounded by the planes

$$x + y + z = 1 \quad \text{and} \quad 2x + 2y + z = 2$$

in the first octant (i.e. where $x, y, z \geq 0$). Let $\mathcal{S}$ be the surface that is the boundary of this region, with outward pointing normal vectors.

(a) (2 points) Calculate the divergence of the vector field

$$\mathbf{F} = \langle xz, 3yz, 2z^2 \rangle.$$

(b) (8 points) Calculate the flux of **F** through the surface $\mathcal{S}$.

2. Consider the region, $\mathcal{P}$, in $\mathbb{R}^2$ bounded by the lines $y = -x$, $y = 3 - x$, $y = 2x$ and $y = 2x - 6$. This is a parallelogram.

(a) (3 points) Find a linear change of coordinates $G(u, v)$ such that G maps the rectangle $[0, 3] \times [0, 6]$ to $\mathcal{P}$. Calculate the Jacobian of G.

(b) (3 points) Calculate the integral $\iint_{\mathcal{P}} x^2 - y^2 \, dA$.

(c) (4 points) Now consider the region $\mathcal{E} \subset \mathbb{R}^3$ bounded by the planes $y = -x$, $y = 3 - x$, $y = 2x$ and $y = 2x - 6$, and by the xy-plane below and the surface $z = 9 + x^2 - y^2$ above. Calculate the volume of $\mathcal{E}$. *Hint: the previous part should be very useful.*

3. (a) (5 points) Consider the region $\mathcal{R}$ of the plane described by $(\sqrt[3]{x})^2 + (\sqrt[3]{(y/8)})^2 \leq 1$. Use Green's theorem to calculate the area of $\mathcal{R}$. **You must use Green's theorem to get full credit!** *Hint: try a parameterisation with $x(t) = \cos^n t$ for some appropriate n. The following formulas might be useful*

$$\int \sin^2 t \, dt = \frac{1}{2}(t - \sin t \cos t) \quad \text{and} \quad \int \sin^n t \, dt = -\frac{1}{n} \sin^{n-1} t \cos t + \frac{n-1}{n} \int \sin^{n-2} t \, dt$$

(b) (5 points) A spiral ramp (pictured low) is 10 feet wide and in one complete 2π revolution it goes up π feet. What is the area of this spiral ramp?

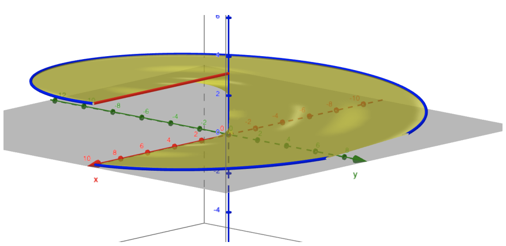

Figure 1: View from above and to the side

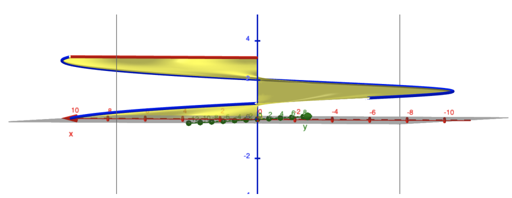

Figure 2: View from the side

4. (a) (2 points) Consider the vector field
$$\mathbf{F} = \langle xy, yz, zx \rangle$$
Calculate the curl of $\mathbf{F}$.

   (b) (8 points) Consider the closed curve $\mathcal{C}$ determined by the intersection of the cylinder $x^2 + y^2 = 1$ and the surface $z = x^2$, with orientation counter clockwise when looking from above. Calculate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

5. Consider the vector field
$$\mathbf{F}(x, y, z) = \left\langle \frac{e^{z^2}}{1 + y^2 + z^2}, \frac{1}{1 + x^4}, z^2 + 1 \right\rangle.$$

   (a) (2 points) What is the divergence of $\mathbf{F}$?

   (b) (8 points) Let $\mathcal{S}$ be the hemipshere $x^2 + y^2 + z^2 = 1$ where $z \geq 0$ with outward pointing orientation. What is $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$? *Hint: Think about the divergence theorem, but be careful, the bottom of the hemisphere is not included in* $\mathcal{S}$*!*

6. (This question is for extra credit points) Consider the vector field
$$\mathbf{F}(x, y, z) = \langle x, \frac{-z}{y^2 + z^2}, \frac{y}{y^2 + z^2} \rangle$$
with domain $\mathbb{R}^3 \setminus \{(x, 0, 0) \mid x \in \mathbb{R}\}$.

   (a) (2 points) Calculate the curl of $\mathbf{F}$.

   (b) (3 points) Is $\mathbf{F}$ conservative? Demonstrate your answer with a calculation.

   (c) (7 points) Consider the oriented curve $\mathcal{C}$ given by the parametrisation
$$\mathbf{r}(t) = (t, \cos 2\pi t, 2 \sin 2\pi t) \text{ where } t \in [0, 2].$$
Calculate the integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. *Hint: so you have an idea of what this curve looks like, it is an elliptic spiral around the* x*-axis.*