Auston Wang

32 B Final

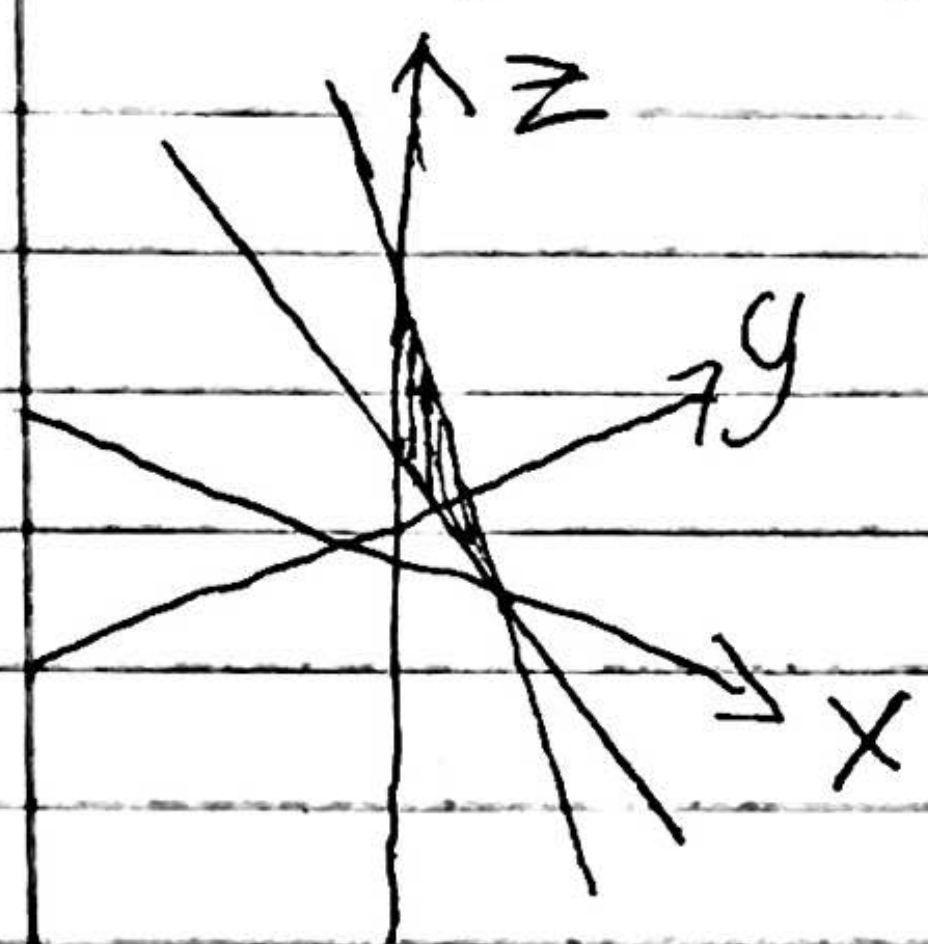
$\boxed{Q1}$ $x+y+z=1$, $2x+2y+z=2$

$x, y, z \geq 0$ $S =$ boundary of region

(a.) $\operatorname{div} F = \frac{d}{dx}(xz) + \frac{d}{dy}(3yz) + \frac{d}{dz}(2z^2)$

$= z + 3z + 4z = 8z$

"diagonal" view



(b.) We can use the divergence formula

$$\iiint_{\mathcal{E}} f \, dV = \iint_{d\mathcal{E}} F \cdot dS$$

where $d\mathcal{E} = S$, and $\mathcal{E}$ is the volume bound by S. (outward normal vectors)

Flux $= \iiint_{\mathcal{E}} 8z \, dV$ $8z = \operatorname{div}(F)$

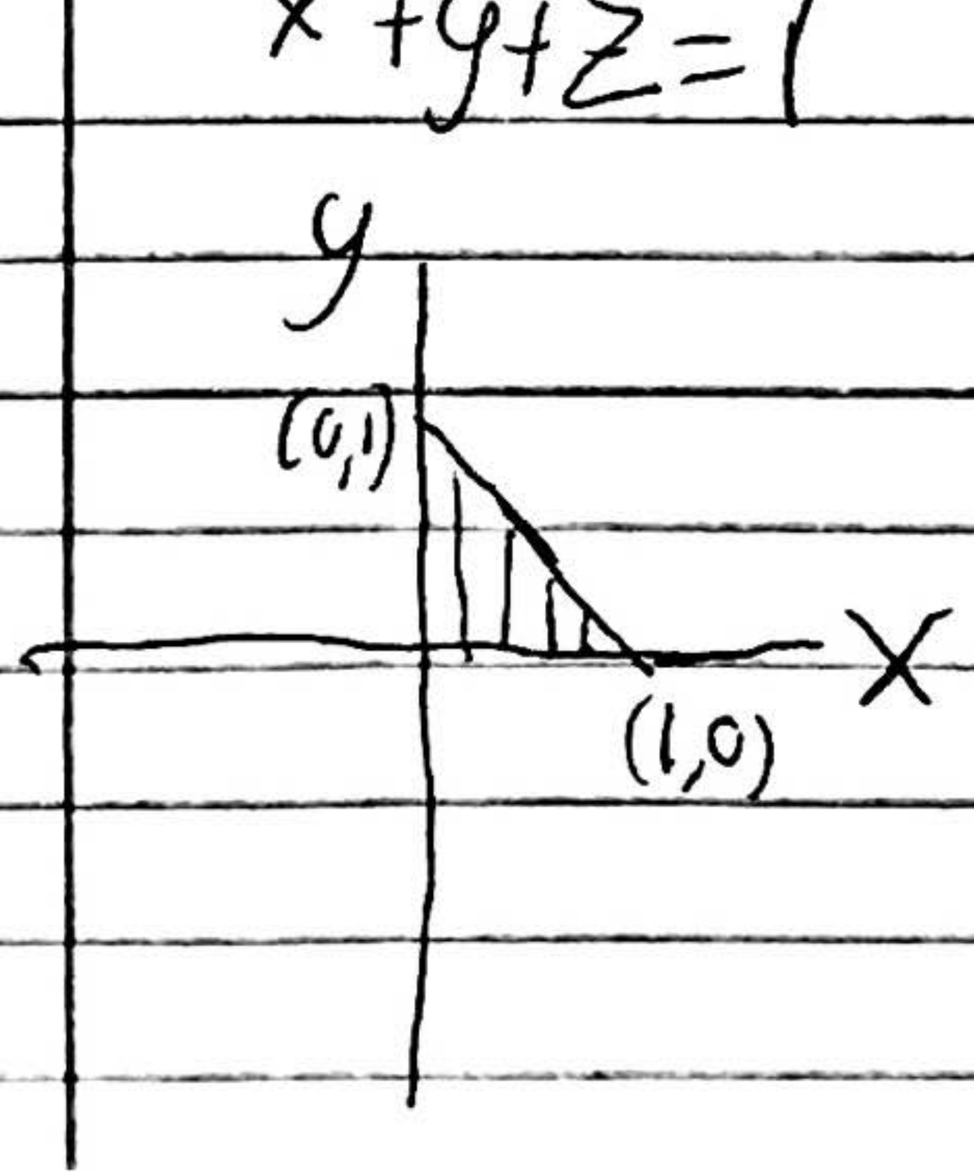
Note our region is z-simple: z upper bound:

$2x+2y+z=2 \implies z = 2(1-x-y)$

z lower bound:

$x+y+z=1 \implies z = (1-x-y)$

Remaining region



Remaining region is y-simple:

$0 \leq y \leq 1-x$

Final integrate dx:

$0 \leq x \leq 1$

Q1 Thus we find our iterated integral

$$\iiint_E 8z\, dV = \int_0^1 \int_0^{1-x} \int_{1-x-y}^{2(1-x-y)} 8z\, dz\, dy\, dx$$

Solve our integral:

$$\int_0^1 \int_0^{1-x} 4z^2 \Big|_{1-x-y}^{2(1-x-y)} dy\,dx = \int_0^1 \int_0^{1-x} 4\left(4(1-x-y)^2 - (1-x-y)^2\right) dy\, dx$$

$$= \int_0^1 \int_0^{1-x} 12(1-x-y)^2\, dy\,dx = \int_0^1 \int_0^{1-x} \left(12x^2 + 24xy + 12y^2 - 24x - 24y + 12\right) dy\, dx$$

$$= \int_0^1 12x^2y + 12xy^2 + 4y^3 - 24xy - 12y^2 + 12y \Big|_0^{1-x} dx$$

$$= \int_0^1 12x^2(1-x) + 12x(1-x)^2 + 4(1-x)^3 - 24x(1-x) - 12(1-x)^2 + 12(1-x)\, dx$$

$$= \int_0^1 12x^2 - 12x^3 + 12x - 24x^2 + 12x^3 - 4x^3 + 12x^2 - 12x + 4 - 24x \ldots$$

$$+ 24x^2 - 12 + 24x - 12x^2 + 12 - 12x\, dx$$ cancel terms

$$= \int_0^1 12x^2 - 4x^3 - 12x + 4\, dx$$

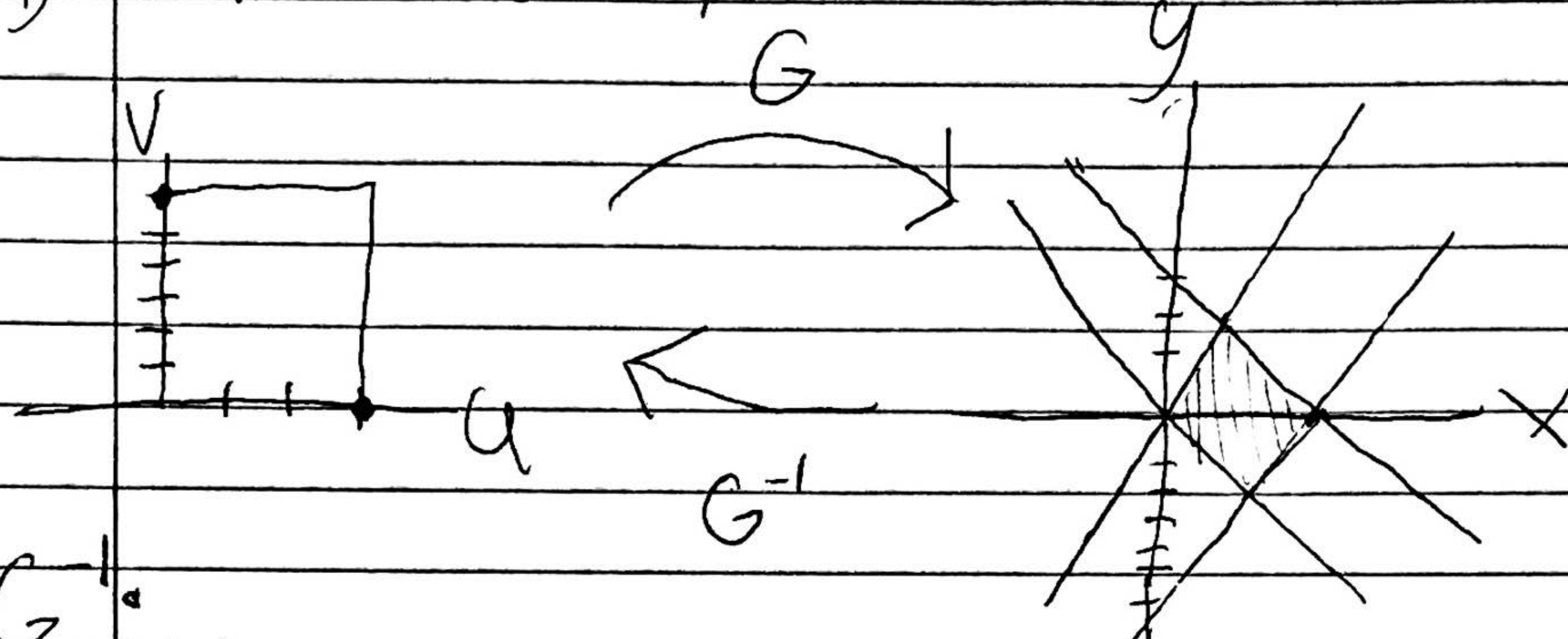
$$= 4x^3 - x^4 - 6x^2 + 4x \Big|_0^1$$

$$= 4 - 1 - 6 + 4 = 1$$

Flux through $S = 1$

$\boxed{Q2}$ $P \in \mathbb{R}^2$: $y=-x$, $y=3-x$, $y=2x$, $y=2x-6$

(a) want to find G;



G^{-1}:

let $u = x+y$ $\quad 0 \le x+y \le 3$

$v = 2x-y$ $\quad 0 \le 2x-y \le 6$

G: Solve for x, y:

$x = u-y$

$2u-2y-y = v \Rightarrow y = \frac{2u}{3} - \frac{v}{3}$

$x = \frac{u}{3} + \frac{v}{3}$

$G(u,v) = \langle \frac{1}{3}u + \frac{1}{3}v, \frac{2}{3}u - \frac{1}{3}v \rangle$

$$\mathrm{Jac}\,G = \det\begin{pmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

$$= -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

$\boxed{Q2}$

(b.) $\iint_P x^2 - y^2 \, dA$, we can use our map:

$D = [0,3] \times [0,6]$

$\iint_D f(G(u,v)) \, |J(G)| \, dA$

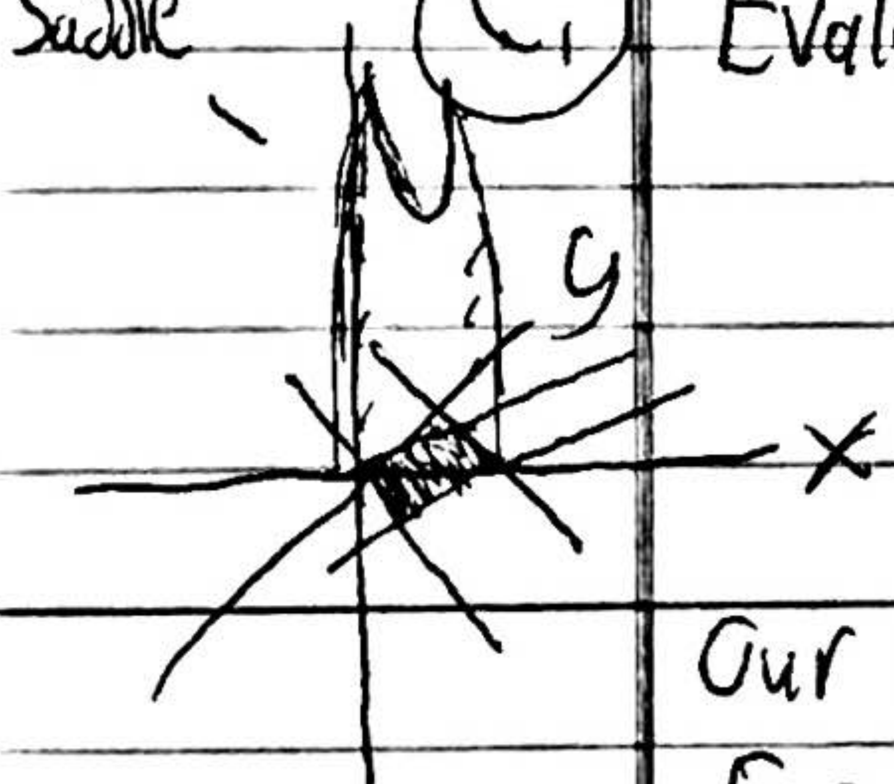
$= \int_0^6 \int_0^3 (\frac{1}{3}u + \frac{1}{3}v)^2 - (\frac{2}{3}u - \frac{1}{3}v)^2 (\frac{1}{3}) \, du \, dv$

$= \frac{1}{3} \int_0^6 \int_0^3 \frac{1}{9}(-3u^2 + 6uv) \, du \, dv$

$= \frac{1}{3} \int_0^6 \frac{1}{9}(-u^3 + 3u^2 v) \Big|_0^3 dv = \frac{1}{3} \int_0^6 -3 + 3v \, dv$

$= \frac{1}{3}(-3v + \frac{3}{2}v^2) \Big|_0^6 = \frac{1}{3}(-18 + 54) = 12$

(C.) Evaluate triple integral: Use map G. $\iiint_E 1 \, dV$



region is z simple:

convert bounds

$0 \leq z \leq 9 + x^2 - y^2$

$0 \leq z \leq 9 + \frac{1}{9}(-3u^2 + 6uv)$

Our remaining region is D

$[0,3] \times [0,6]$

$= \int_0^6 \int_0^3 \int_0^{9 + \frac{1}{9}(-3u^2 + 6uv)} 1 \cdot \frac{1}{3} \, dz \, du \, dv$

Q2

C. Cont...

note this was computed in part b.

$$=\frac{1}{3}\int_0^6\int_0^3 9+\frac{1}{9}(-3u^2+6uv)\,du\,dv$$

$$=\frac{1}{3}\int_0^6\int_0^3 9\,du\,dv+\frac{1}{3}\int_0^6\int_0^3\frac{1}{9}(-3u^2+6uv)\,du\,dv$$

as computed in b

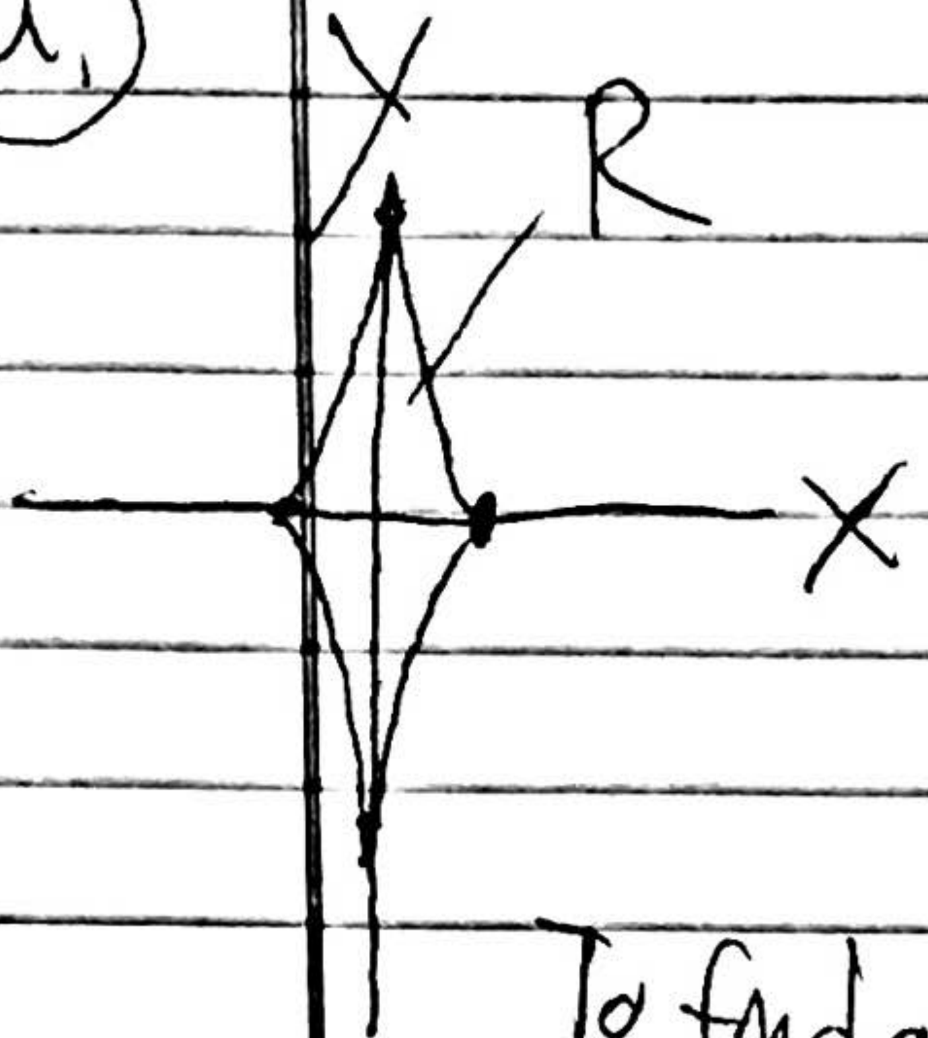
$$\frac{1}{3}(9\cdot3\cdot6)=54 \quad + \quad 12$$

$$54+12=66$$

thus the volume of $E = 66$

$\boxed{Q3}$ $R: \; x^{2/3} + \left(\frac{y}{8}\right)^{2/3} \le 1$

(a)



Note we can calculate the area with Green's theorem

$$\iint_D f \, dA = \int_{\partial D} F \cdot dr$$

To find area, we want $\iint_R 1 \, dA$, thus let us find an F s.t. $\text{curl}(F) = 1$ let us use $F = \cancel{\langle 0, x \rangle}$ $\langle -y, 0 \rangle$

Next we need to parameterize the boundary:

guess with $a\cos^n t,\ b\sin^n t$: ultimately we end up with

$$r(t) = (\cos^3 t, \, 8\sin^3 t) \qquad 0 \le t \le 2\pi$$

$$r'(t) = (-3\cos^2 t \sin t, \, 24 \sin^2 t \cos t)$$

$$\int_{\partial R} F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t) \, dt$$

$$= \int_0^{2\pi} (-8\sin^3 t, 0) \cdot (-3\cos^2 t \sin t, 24 \sin^2 t \cos t) \, dt$$

$$= \int_0^{2\pi} 24 \sin^4 t \cos^2 t \, dt \qquad \text{Sub: } \cos^2 t = 1 - \sin^2 t$$

$$= \int_0^{2\pi} 24 \sin^4 t - 24 \sin^6 t \, dt$$

Q3 We are given $\int \sin^2 t \, dt = \frac{1}{2}(t - \sin t \cos t)$

(a) Cont...

$$\int \sin^n t \, dt = -\frac{1}{n}\sin^{n-1} t \cos t + \frac{n-1}{n}\int \sin^{n-2} t \, dt$$

First compute $24\int_0^{2\pi} \sin^4 t \, dt$

$$= 24\left(-\frac{1}{4}\sin^3 t \cos t + \frac{3}{4}\cdot\underbrace{\frac{1}{2}(t - \sin t \cos t)}_{\int \sin^2 t \, dt}\right)\Bigg|_0^{2\pi}$$

$$= 24\left(0 + \frac{3}{8}\cdot 2\pi + 0\right) = 24\cdot\frac{3}{4}\pi = 18\pi$$

Next calculate $24\int \sin^6 t \, dt$ — Solve this recursively

$$= 24\left(-\frac{1}{6}\sin^5 t \cos t + \frac{5}{6}\underbrace{\left(-\frac{1}{4}\sin^3 t + \frac{3}{4}\cdot\frac{1}{2}(t - \sin t \cos t)\right)}_{\int_0^{2\pi} \sin^4 t}\right)\Bigg|_0^{2\pi}$$

reuse our result

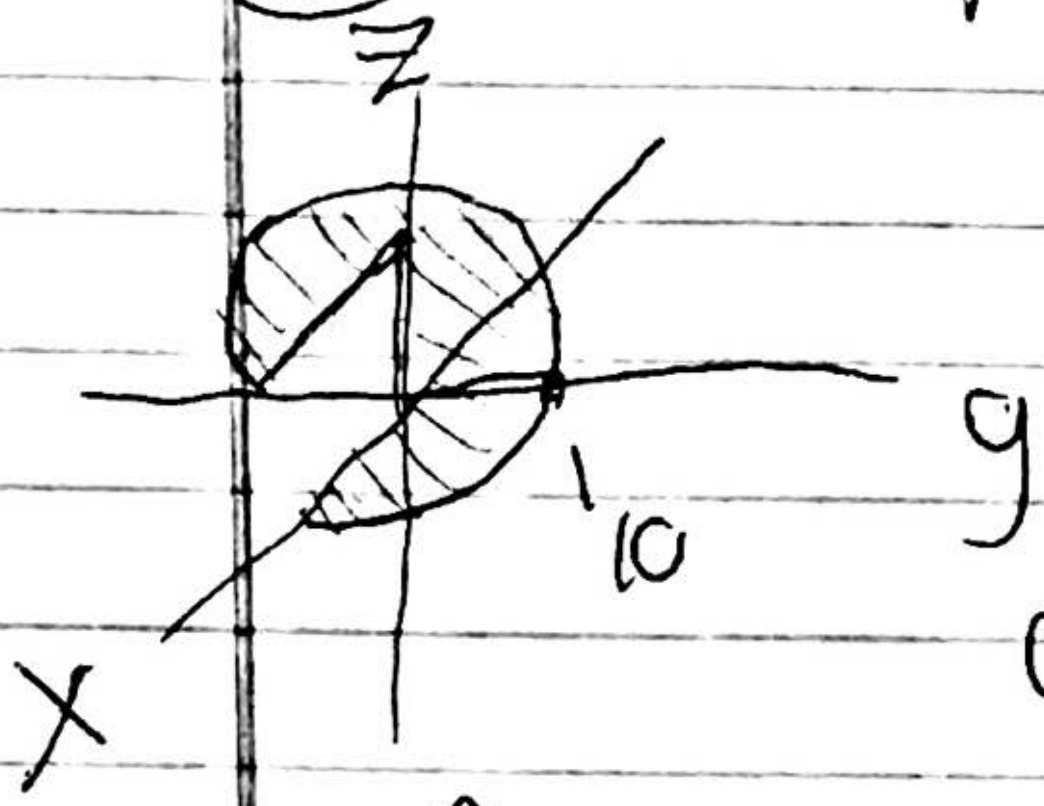
$$= \left(0 + \frac{5}{6}(18\pi)\right) = 15\pi$$

$$24\int_0^{2\pi} \sin^4 t - \sin^6 t \, dt = 18\pi - 15\pi = 3\pi$$

thus $\iint_R 1 \, dA = 3\pi$ and by Greens theorem we get our area.

Q3

(b) Let us parameterize our surface, this essentially represents a disk with a z component.



It takes a full revolution to go up π feet

Let $G(r,\theta) = (r\cos\theta, r\sin\theta, \frac{\theta}{2})$
where $r \in [0,10]$, $\theta \in [0, 2\pi]$

$D = [0,10] \times [0, 2\pi]$

We want to calculate surface area or $\iint_S 1 \, dS$

$T_r = (\cos\theta, \sin\theta, 0)$ $\quad N = T_r \times T_\theta$

$T_\theta = (-r\sin\theta, r\cos\theta, \frac{1}{2})$

orientation doesn't matter
We only want surface area.

$N = (\frac{1}{2}\sin\theta, -\frac{1}{2}\cos\theta, r)$

$|N| = \sqrt{\frac{1}{4}(\sin^2\theta + \cos^2\theta) + r^2} = \sqrt{\frac{1}{4} + r^2}$

$\iint_S 1 \, dS = \int_0^{2\pi}\int_0^{10} 1 \cdot \sqrt{\frac{1}{4} + r^2} \, dr \, d\theta$ (u sub)

$u = 2r$
$du = 2\,dr$

$= \int_0^{2\pi}\int_0^{20} \frac{1}{2}\sqrt{\frac{1}{4} + \frac{1}{4}u^2} \, du \, d\theta = \int_0^{2\pi}\int_0^{20} \frac{1}{2}\cdot\frac{1}{2}\sqrt{1+u^2} \, du \, d\theta$

use hint from prob 3

$= \int_0^{2\pi} \frac{1}{2}\cdot\frac{1}{4}\left(u\sqrt{u^2+1} + \sinh^{-1}(u)\right)\Big|_0^{20} d\theta$

$= \frac{\pi}{4}\left(20\sqrt{401} + \sinh^{-1}(20)\right) \cong 317.449$

$\boxed{Q4}$ $F = \langle xy, yz, zx \rangle$

(a) $\nabla \times F = \begin{matrix} d_x & xy \\ d_y & yz \\ d_z & zx \end{matrix} = \left\langle \frac{d}{dy}(zx) - \frac{d}{dz}(yz), \frac{d}{dz}(xy) - \frac{d}{dx}(zx), \right.$

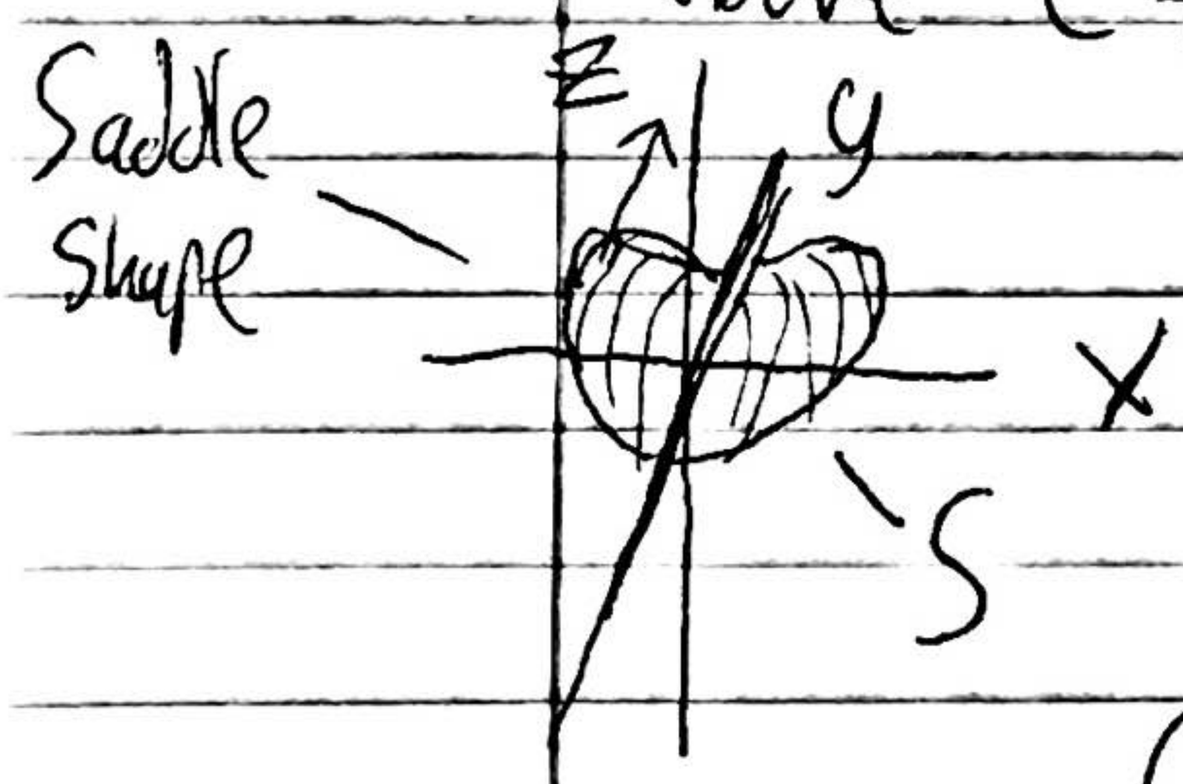
$$\left. \frac{d}{dx}(yz) - \frac{d}{dy}(xy) \right\rangle$$

$\nabla \times F = \langle -y, -z, -x \rangle$

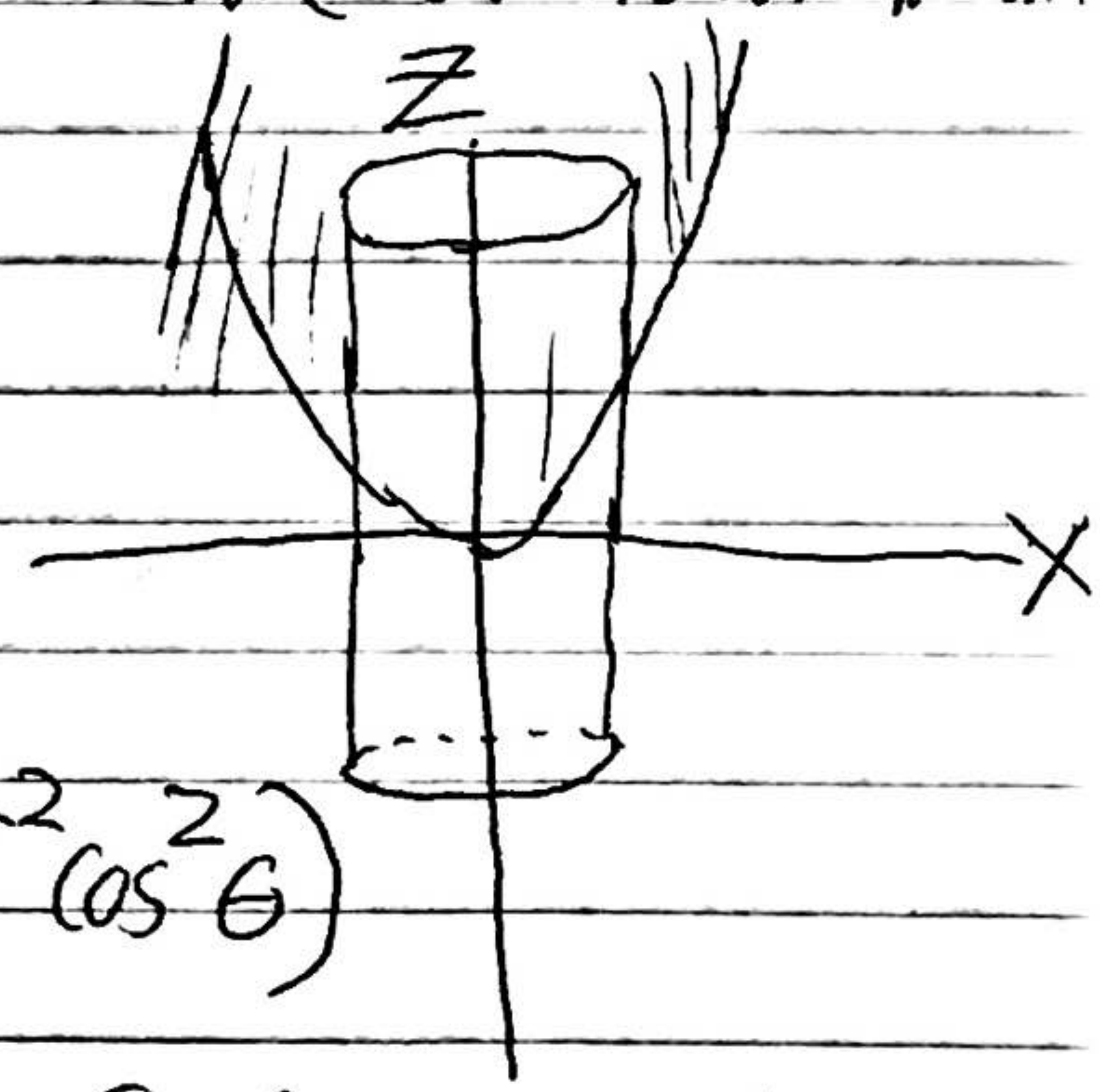
(b) Let us use Stokes theorem: $\iint_S F \cdot ds = \int_{\partial S} A \cdot dr$

where ∂S, our curve boundary, is C.

Thus since C is CCW from above, S will have outward normal from above ($+z$ direction)



x, y can be parameterized like a disc, we also find $z = x^2$, thus:



$$G(r, \theta) = (r\cos\theta, r\sin\theta, r^2\cos^2\theta)$$

$T_r = (\cos\theta, \sin\theta, 2r\cos^2\theta)$ $0 \leq \theta \leq 2\pi$

$T_\theta = (-r\sin\theta, r\cos\theta, -2r^2\cos\theta\sin\theta)$ $0 \leq r \leq 1$

$N = T_r \times T_\theta = (-2r^2\cos\theta, 0, r)$ — used $1 - \sin^2\theta = \cos^2\theta$
$1 - \cos^2\theta = \sin^2\theta$

$\iint_S f(G(r,\theta)) \cdot N \, ds = \iint_S f \cdot ds$ where $f = \text{curl}(F)$

Q4

b. cont...

$$= \int_0^{2\pi}\int_0^1 (-r\sin\theta, -r^2\cos^2\theta, -r\cos\theta)\cdot(-2r^2\cos\theta, 0, r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r^3\sin\theta\cos\theta - r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\sin 2\theta - r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}r^4\sin 2\theta - \frac{1}{3}r^3\cos\theta \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}\sin 2\theta - \frac{1}{3}\cos\theta\,d\theta$$

$$= -\frac{1}{8}\cos 2\theta + \frac{1}{3}\sin\theta \Big|_0^{2\pi} = 0+0+0+0$$

$$= 0$$

thus By stokes theorem

$$\iint_C F\cdot dr = 0$$

Q5

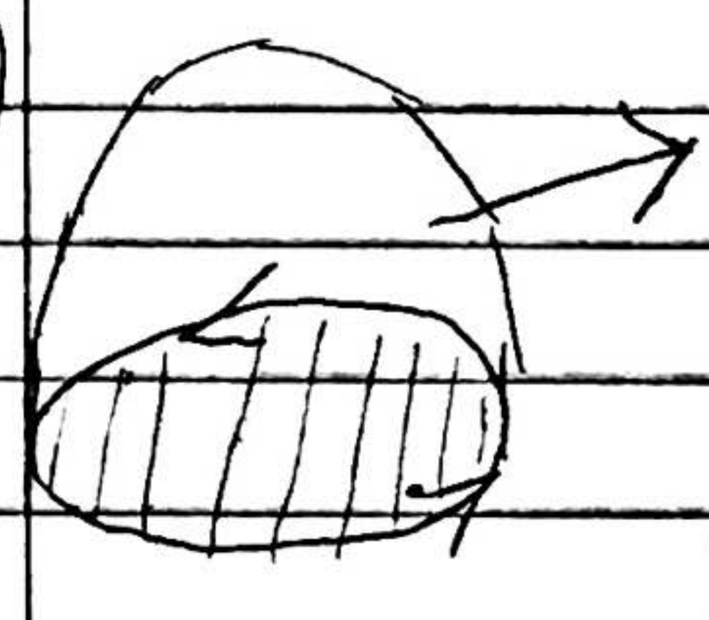
(a) $F(x,y,z) = \left\langle \dfrac{e^{z^2}}{1+y^2+z^2}, \dfrac{1}{1+x^4}, z^2+1 \right\rangle$

$\operatorname{div}(F) = \dfrac{d}{dx}\left(\dfrac{e^{z^2}}{1+y^2+z^2}\right) + \dfrac{d}{dy}\left(\dfrac{1}{1+x^4}\right) + \dfrac{d}{dz}\left(z^2+1\right)$

$= 0 + 0 + 2z = 2z$

(b)

$S \rightarrow$



Surface S: $x^2+y^2+z^2=1$, $z \geq 0$

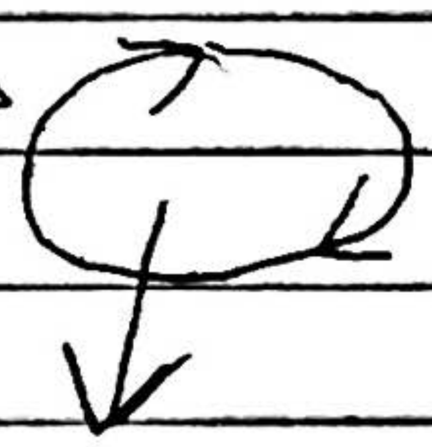
Let us use divergence theorem

$\iint_S F \cdot dS = \iiint_E \operatorname{div}(F)\, dV$

however our Surface S is not closed.

let T = unit circle at $(0,0)$ with outward normal ($-z$ direction)

$T \rightarrow$



If we add Surface S and T, we get a closed surface that bounds the hemisphere E.

then $\iint_{S+T} F \cdot ds = \iiint_E 2z\, dV$

$\iint_{S+T} F \cdot ds = \iint_S F \cdot ds + \iint_T F \cdot ds$

$\iint_S F \cdot ds = \iiint_E 2z\, dV - \iint_T F \cdot ds$

Q5 (b.) cont...

First calculate $\iint_{\mathcal{T}} F \cdot ds$

Parameterize disk T: Surface $\mathcal{T}$

$G(\theta, r) = (r\cos\theta, r\sin\theta, 0)$

where: $0 \le \theta \le \pi$, $0 \le r \le 1$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$ $N = T_\theta \times T_r$

$T_r = (\cos\theta, \sin\theta, 0)$

$N = (0, 0, -r)$ — Correct orientation, points outwards.

$$\iint_{\mathcal{T}} F \cdot ds = \int_0^{2\pi}\int_0^1 \left(\frac{e^0}{1 + r^3\sin^2\theta + 0}, \frac{1}{1+(r\cos\theta)^4}, 0+1\right) \cdot (0, 0, -r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -r\, dr\, d\theta$$

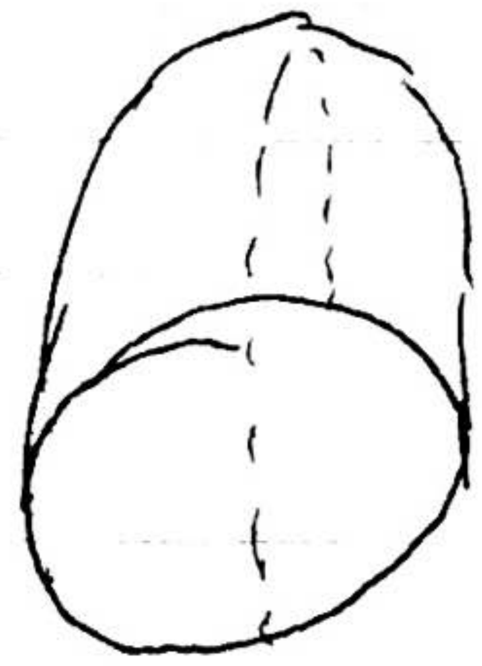
$$= \int_0^{2\pi} -\frac{1}{2}r^2\Big|_0^1 d\theta = \int_0^{2\pi} -\frac{1}{2}\, d\theta$$

$$= -\frac{1}{2}\theta\Big|_0^{2\pi} = -\pi = \iint_{\mathcal{T}} F \cdot ds$$

Next we must find $\iiint_{\mathcal{E}} 2z\, dV = \iint_{S+\mathcal{T}} F \cdot ds$

$\boxed{Q5}$ (b) cont...

region E



Let us use Spherical coordinates for our triple integral

$$G(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$

$$J(G) = \rho^2 \sin\phi$$

$$\iiint_E 2z \, dV \Rightarrow \iiint_E 2\rho \cos\phi \cdot \rho^2 \sin\phi \, dV_{\rho\theta\phi}$$

Note our bounds for the Sphere: $0 \leq \rho \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/2$ since hemisphere

$$= \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 2\rho^3 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \frac{1}{4} \cdot \sin 2\phi \, d\theta \, d\phi = \int_0^{\pi/2} \frac{\pi}{2} \cdot \sin 2\phi \, d\phi$$

$$= -\frac{\pi}{4} \cos 2\phi \Big|_0^{\pi/2} = \frac{\pi}{2\cancel{4}} = \iint_{S+T} F \cdot dS$$

thus

$$\iint_S F \cdot dS = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$$

$\boxed{Q6}$ (a) $F = \left\langle x, \frac{-z}{y^2+z^2}, \frac{y}{y^2+z^2} \right\rangle$

$$\text{Curl } F = \left\langle \frac{d}{dy}\left(\frac{y}{y^2+z^2}\right) - \frac{d}{dz}\left(\frac{-z}{y^2+z^2}\right), \frac{d}{dz}(x) - \frac{d}{dx}\left(\frac{y}{y^2+z^2}\right), \frac{d}{dx}\left(\frac{-z}{y^2+z^2}\right) - \frac{d}{dy}(x) \right\rangle$$

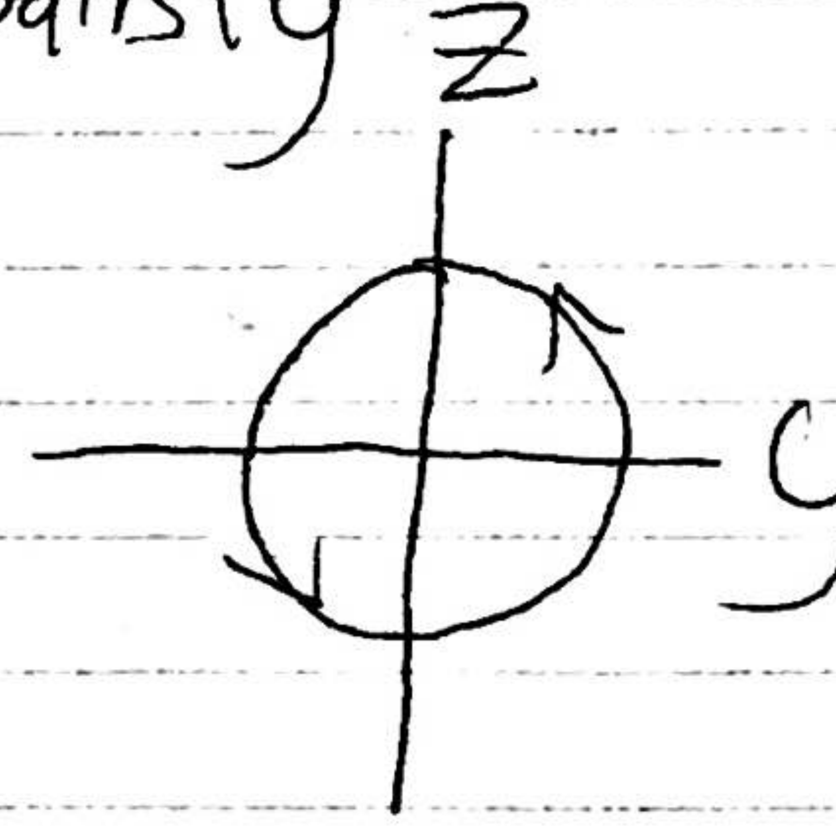
Quotient rule $= \left\langle \frac{\cancel{y^2}+\cancel{z^2} - \cancel{2y(y)}}{(y^2+z^2)^2} + \frac{\cancel{y^2}+\cancel{z^2} - \cancel{2z(z)}}{(y^2+z^2)^2}, 0, 0 \right\rangle$

$= \langle 0, 0, 0 \rangle$ Curl $F = 0$

(b) If F is conservative on $\mathbb{R}^3 \setminus \{x, 0, 0 \mid x \in \mathbb{R}\}$

then any closed loop C should satisfy

$\oint_C F \cdot dr = 0$



Let C = unit circle on yz plane

Parameterize C:

$\boxed{0 \le t \le 2\pi}$ $r(t) = (0, \cos t, \sin t)$

$r'(t) = (0, -\sin t, \cos t)$

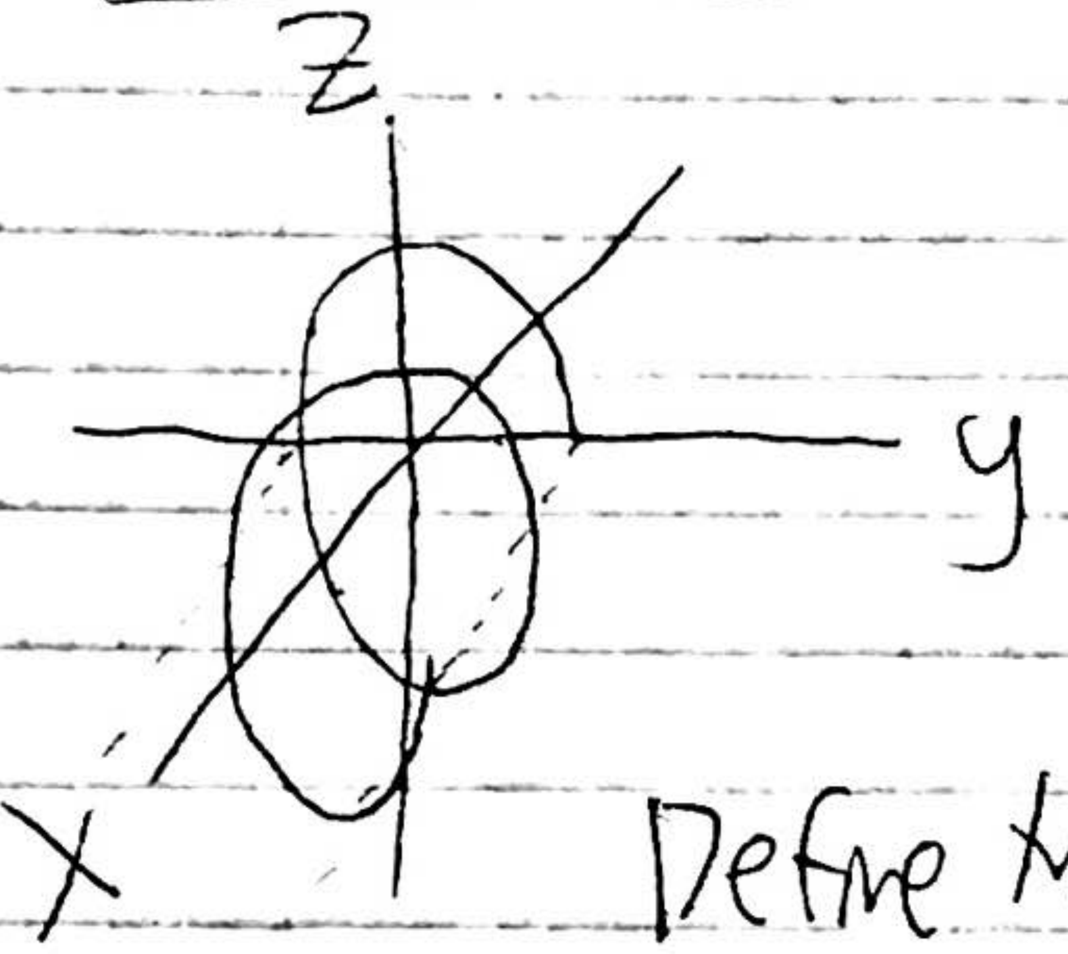
$\oint_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$

$= \int_0^{2\pi} \left(0, \frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t}\right) \cdot (0, -\sin t, \cos t)\, dt$

$= \int_0^{2\pi} \sin^2 t + \cos^2 t = t \Big|_0^{2\pi} = 2\pi \neq 0$

thus F is <u>not</u> conservative on this domain.

$\boxed{Q6}$ (C.) $r(t) = (t, \cos 2\pi t, 2\sin 2\pi t)$ $t \in [0,2]$



Note that calculating the integral $\int_C F \cdot dr$ is very difficult.

Let us try to use path independence

Define two simply connected domains, ($\nabla \times F = 0$)

D_+: $z \geq 0$ $- x\text{-axis}$ $\leftarrow \setminus \{(x,0,0) \mid x \in \mathbb{R}\}$

D_-: $z \leq 0 - x\text{-axis}$

Let us separate C into 4 separate curves:

$2\sin 2\pi t = 0$ at $t = 0, 0.5, 1, 1.5, 2$

let C_1 be C from $t \in [0, 0.5]$

C_2 be C $t \in [0.5, 1]$

C_3 be C $t \in [1, 1.5]$

C_4 be C $t \in [1.5, 2]$

$\int_C F \cdot dr =$

$\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$

$(F \cdot dr)$

C_1 and C_3 are contained in D_+

C_2 and C_4 are contained in D_-

thus we can use path independence 4 times.

let's parameterize an easier curve that hits all start/end points

$r(t) = (t, \cos 2\pi t, \sin 2\pi t)$ — non elliptical spiral around x-axis

$r'(t) = (1, -2\pi \sin 2\pi t, 2\pi \cos 2\pi t)$

$$F = \left\langle x, \frac{-z}{y^2+z^2}, \frac{y}{y^2+z^2} \right\rangle$$

$\boxed{Q6}$ C_1 cont... Let us use $r(t)$ to find independent paths for $C_1\ C_2\ C_3\ C_4$

let $r(t)$ define curve A

$$\int_A F \cdot dr = \int_0^{2} \left(t, \frac{-\sin 2\pi t}{1}, \frac{\cos 2\pi t}{1}\right) \cdot (1, -2\pi \sin 2\pi t, 2\pi \cos 2\pi t)$$

$$= \int_0^{2} t + \frac{2\pi(\sin^2 2\pi t + \cos^2 2\pi t)}{1}\, dt = \int_0^2 t + 2\pi\, dt$$

Note we can split curve A into 4 parts s.t. each part corresponds with a part of C

$$\int_A F \cdot dr = \int_0^{0.5} t + 2\pi\, dt + \int_{0.5}^{1} t + 2\pi\, dt + \int_1^{1.5} t + 2\pi\, dt + \int_{1.5}^{2} t + 2\pi\, dt$$

$$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow \qquad \Updownarrow$$

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr + \int_{C_3} F \cdot dr + \int_{C_4} F \cdot dr$$

Same start points / end points for all 4 pairs of curves,

thus we find that if we sum the curves

$$\int_C F \cdot dr = \int_A F \cdot dr = \int_0^2 t + 2\pi\, dt$$

$$= \frac{1}{2}t^2 + 2\pi t \Big|_0^2 = 2 + 4\pi$$

Note: Cannot just directly use $\int_A F \cdot dr$ due to lack of common simply connected domain.

$$\int_C F \cdot dr = 2 + 4\pi$$