Math 32B Exam 2

TOTAL POINTS

44 / 50

QUESTION 1

Vortex Field 10 pts

1.1 Angle 4 / 4

✓ + **2 pts** **Correct setup (parametrization and integral)**

✓ + **2 pts** **Correct evaluation of integral**

+ **4 pts** Using potential function arctan(y/x)

- **1 pts** Minor error

+ **0 pts** Incorrect

+ **1 pts** Attempting to parametrize using sines and cosines

1.2 Ellipse 3 / 3

✓ - **0 pts** **Reasonable attempt**

- **3 pts** No reasonable attempt

1.3 Conservative 2 / 3

✓ + **2 pts** **No**

+ **2 pts** Reasoning

+ **1 pts** Incorrect

+ **1 pts** Partially correct reasoning

QUESTION 2

Integrals 10 pts

2.1 Line integral 2 / 5

+ **5 pts** Correct

+ **0 pts** Click here to replace this description.

✓ + **2 pts** **Set up an integral with a correct parameterization**

+ **2 pts** applied fundamental theorem

+ **2 pts** found a potential function

+ **1 pts** Correct answer

2.2 Surface Integral 5 / 5

✓ + **5 pts** **Correct**

+ **2 pts** Correct | normal vector |

+ **1 pts** Correct orientation on normal

+ **2 pts** Correct bounds

+ **0 pts** Click here to replace this description.

QUESTION 3

Parametrizations 10 pts

3.1 Triangle 5 / 5

✓ + **1 pts** **Correct inequality in D independent of the other variable.**

✓ + **2 pts** **Correct inequality in D dependent on the other variable.**

✓ + **2 pts** **Correct parameterisation of G.**

+ **0 pts** Incorrect

3.2 Cylinder 3 / 5

✓ + **1 pts** **Correct u bounds.**

✓ + **2 pts** **Correct v bounds.**

+ **2 pts** Correct choice (negative).

+ **0 pts** Incorrect.

QUESTION 4

Simply connected 3 pts

4.1 a 1 / 1

- **1 pts** Always True

✓ - **0 pts** **Only if D is simply connected**

- **1 pts** No answer

4.2 b 1 / 1

✓ - **0 pts** **Always True**

- **1 pts** Only if D is SC

- **0 pts** No answer

4.3 c 1 / 1

✓ - 0 pts **Always True**

- 1 pts Only true if D is SC

- 1 pts No answer

QUESTION 5

Conservative Vector Fields 5 pts

5.1 a 1 / 1

- 1 pts All

✓ - 0 pts **Conservative**

5.2 b 1 / 1

- 1 pts All

✓ - 0 pts **Conservative**

5.3 c 1 / 1

✓ - 0 pts **All**

- 1 pts Conservative

5.4 d 0 / 1

- 0 pts All

✓ - 1 pts **Conservative**

- 1 pts No Answer

5.5 e 1 / 1

✓ - 0 pts **All**

- 1 pts Conservative

- 1 pts No Answer

QUESTION 6

MC 12 pts

6.1 a 3 / 3

- 3 pts Yes

✓ - 0 pts **No**

- 3 pts Not enough info

- 3 pts No answer

6.2 b 3 / 3

✓ - 0 pts **fifth choice**

- 3 pts any other choice

6.3 c 3 / 3

✓ + 1 pts **first is zero (max of 3 points for this problem)**

✓ + 1 pts **second is negative**

✓ + 1 pts **third is zero**

✓ + 1 pts **fourth is positive**

- 1 pts too many zeros, negatives, or positives

+ 0 pts no answer

6.4 d 3 / 3

✓ - 0 pts **F is tangent to S**

- 3 pts F is perpendicular to S

- 3 pts No answer

QUESTION 7

7 Bonus 1 / 0

✓ + 1 pts **Bonus point**

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- Fill out your name, section letter, and UID above.
- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.

Spherical coordinates:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

This derivative might be useful:

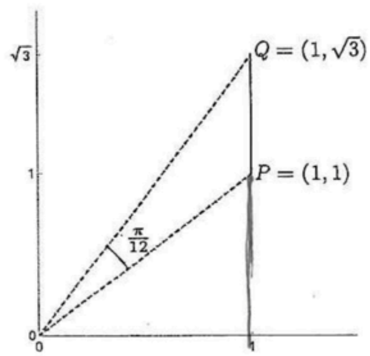
$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

Page:	1	2	3	4	5	Total
Points:	10	10	10	8	12	50
Score:						

You may use this page for scratch work.

1. (10 points) Let $\boldsymbol{F}$ denote the vortex field $\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

(a) Let C be the straight line segment from $P = (1,1)$ to $Q = (1,\sqrt{3})$ (see the picture). Show that $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \frac{\pi}{12}$.



$\vec{r}(t) = (1, t) \quad 1 \le t \le \sqrt{3}$

$\vec{r}\,'(t) = (0, 1)$

$\vec{F}(\vec{r}(t)) = \left(\frac{t}{1+t^2}, \frac{1}{1+t^2} \right)$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \frac{1}{1+t^2}$

$$\int_1^{\sqrt{3}} \frac{1}{1+t^2}\,dt = (\arctan t)\Big|_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$$

(b) Suppose that C is the ellipse parametrized by $\boldsymbol{r}(t) = \langle 5\cos(2t), 2\sin(2t)\rangle$ for $0 \le t \le \pi$. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$. $\boxed{\text{Box your answer.}}$

$2 \therefore (2\sin t \cos t)^2$
$+ 4 \sin^2 t \cos^2 t$

$\vec{r}\,'(t) = (-5\sin 2t\,(2), 2\cos 2t\,(2))$

$$\vec{F}(\vec{r}(t)) = \left(\frac{-2\sin(2t)}{5^2\cos^2(2t) + 4\sin^2 2t}, \frac{5\cos 2t}{25\cos^2 2t + 4\sin^2 2t} \right)$$

$\sin 2a = 2\sin a \cos a$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \frac{-2\sin 2t(-10\sin 2t) + (4\cos 2t)(5\cos 2t)}{25\cos^2(2t) + 4\sin^2 2t} = \frac{20\sin^2 2t + 20\cos^2 2t}{25\cos^2(2t) + 4\sin^2 2t}$$

$$\int_C \vec{F} \cdot d\vec{r} = \boxed{\int_0^{\pi} \frac{20(\sin^2 2t + \cos^2 2t)}{25\cos^2(2t) + 4\sin^2(2t)}\,dt}$$

(c) Is the vortex field $\boldsymbol{F}$ conservative on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$? Explain your reasoning.

$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}$

$$\frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$\frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}$$

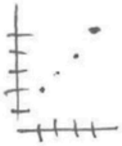
No, the ~~cross~~ cross partials condition does not hold so the curl $(\vec{F}) \neq 0$, therefore it is not conservative.

2. (10 points) You do not need to simplify your answers.

(a) Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where C is the straight line from $P = (1,2,3)$ to $Q = (4,5,6)$ and

$$\boldsymbol{F} = \left\langle \frac{2xy}{x^2+z}, \ln(x^2+z), \frac{y}{x^2+z} \right\rangle.$$

Box your answer



$(1,2)$

$\vec{r}(t) = (t, t+1, t+2)$

$1 \le t \le 4$

$\vec{r}'(t) = (1,1,1)$

$$\vec{F}(\vec{r}(t)) = \left(\frac{2t(t+1)}{t^2+(t+2)}, \ln(t^2+t+2), \frac{(t+1)}{t^2+t+2} \right)$$

$$\int_1^4 \frac{2t^2+2t}{t^2+t+2} + \frac{t+1}{t^2+t+2} + \ln(t^2+t+2) \, dt$$

$$(2t+1)(t+1) = \int_1^4 \frac{2t^2+3t+1}{t^2+t+2} + \ln(t^2+t+2) \, dt$$

$u = t^2+t+2$

$du = (2t+1) \, dt$

$\left[(t^2+t+2)\ln(t^2+t+2) - (t^2+t+2) \right] (2t+1)$

(b) Suppose that $\boldsymbol{F} = \langle y, z, 0 \rangle$ and that $\mathcal{S}$ is the surface parametrized by

$$G(u,v) = (u^3 - v, u+v, v^2), \qquad 0 \le u \le 2, \quad 0 \le v \le 3.$$

with downward-pointing normal.

Fill in the limits and integrand of the integral below so that it equals $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$.

$\vec{T}_u = (3u^2, 1, 0)$

$\vec{T}_v = (-1, 1, 2v)$

$\vec{T}_u \times \vec{T}_v = (2v, -(3u^2 2v), 3u^2 - (-1))$

$= (2v, -3u^2 2v, 3u^2) \Rightarrow$ downward pointing $\quad \vec{N} = (-2v, 3u^2 2v, -3u^2)$

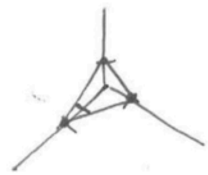
$\vec{F}(G(u,v)) = (u+v, v^2, 0)$

$\vec{F}(G(u,v)) \cdot \vec{N} = (-2v(u+v) + 3u^2 2v(v^2) + 0)$

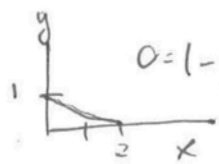
$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_0^3 \int_0^2 -2v(u+v) + 3u^2 2v^3 \, du \, dv$$

You do not need to show work on this page.

3. (10 points) (a) Give a parametrization $G : D \to S$, where S is the triangle in $\mathbb{R}^3$ with vertices $(2,0,0)$, $(0,1,0)$, $(0,0,1)$ in the plane $\frac{1}{2}x + y + z = 1$. Be sure to explicitly specify the domain D and call your parameters u and v.



$z = 1 - \frac{1}{2}x - y$



$0 = 1 - \frac{1}{2}x - y$

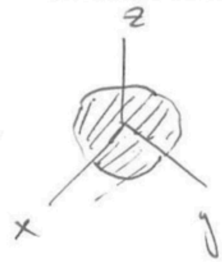
$y = 1 - \frac{1}{2}x$

$$D = \left\{ \quad 0 \le u \le 2 \quad , \quad 0 \le v \le 1 - \tfrac{1}{2}u \quad \right\}$$

$$G(u,v) = \left\langle \quad u \quad , \quad v \quad , \quad 1 - \tfrac{1}{2}u - v \quad \right\rangle$$

(b) Let S be the portion of the cylinder $x^2 + y^2 = 1$ between the xy-plane and the plane $x + z = 2$. We parametrize S by $G(u,v) = \langle \cos u, \sin u, v \rangle$, where (fill these in)

$$0 \le u \le 2\pi \quad , \quad 0 \le v \le 2 - \cos u$$



$z = 2 - x$

The surface integral $\iint_S x \, dS$ is (circle one) negative **zero** positive.

$$\int_0^{2\pi} \int_0^{2 - \cos u} \cos u \, dv \, du$$

$$-(\sin u)\Big|_0^{2\pi} = -0$$

Circle the correct answers.

4. (3 points) Let $\boldsymbol{F}$ be a vector field on an open connected domain $\mathcal{D}$ with continuous second order partial derivatives. Which of the following statements are always true, and which are only true if $\mathcal{D}$ is simply connected?

   (a) If $\text{curl}(\boldsymbol{F}) = 0$ then $\boldsymbol{F}$ is conservative.

   Always true — Only true if $\mathcal{D}$ is simply connected

   (b) If $\boldsymbol{F}$ has a potential function then $\boldsymbol{F}$ is conservative.

   Always true — Only true if $\mathcal{D}$ is simply connected

   (c) If $\boldsymbol{F}$ is conservative then $\text{curl}(\boldsymbol{F})$ is zero.

   Always true — Only true if $\mathcal{D}$ is simply connected

5. (5 points) Which of the following statements is true for all vector fields, and which is true only for conservative vector fields?

   (a) The line integral along a path from P to Q does not depend on which path is chosen.

   True for all vector fields — Only true for conservative vector fields

   (b) The line integral around a closed curve is zero.

   True for all vector fields — Only true for conservative vector fields

   (c) The line integral over an oriented curve $\mathcal{C}$ does not depend on how $\mathcal{C}$ is parametrized as long as each parametrization preserves the orientation of $\mathcal{C}$.

   True for all vector fields — Only true for conservative vector fields

   (d) The (vector) line integral is equal to the (scalar) line integral of the tangential component along the curve. ?

   $\vec{F} \cdot \vec{r}'(t)$ True for all vector fields — Only true for conservative vector fields

   $f(x,y,t)$

   (e) The line integral changes sign if the orientation of the curve is reversed.

   True for all vector fields — Only true for conservative vector fields

$\frac{\partial F}{\partial z}$ cont+ x
~~derivative~~

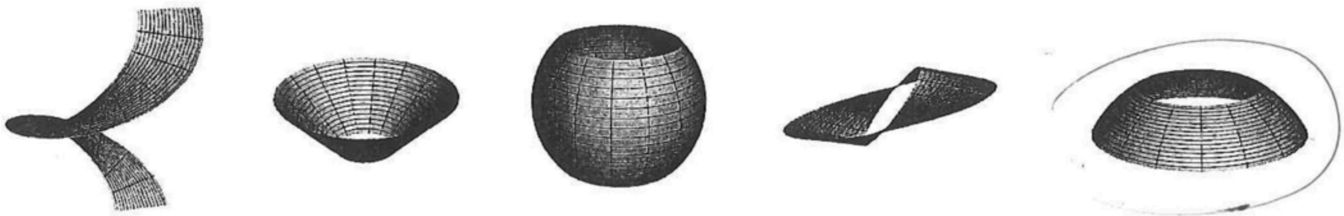
6. (12 points) Multiple choice. Circle the correct answer.

(a) Consider the vector field $\boldsymbol{F} = \langle xz, e^z - yz, \cos x\rangle$. Is there a function f such that $\boldsymbol{F} = \nabla f$?

Yes   No   We don't have enough information

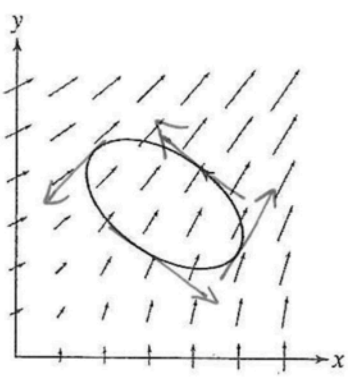
$z\frac{x^2}{2} + H(y,z) = \; - z\frac{y^2}{2} + G(x,z) =$

(b) $G(\theta, \phi) = \langle \cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$ with $0 \le \theta \le 2\pi$ and $\frac{\pi}{6} \le \phi \le \frac{\pi}{3}$ parametrizes which of the surfaces below? (circle one)

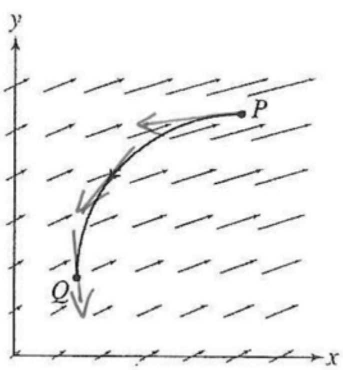


(c) Consider the line integrals $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for the vector fields $\boldsymbol{F}$ and paths $\boldsymbol{r}$ below.
Exactly **two** of the line integrals are zero, **one** is positive, and the remaining **one** is negative. Circle "negative", "zero", or "positive" below each picture to indicate your answers.

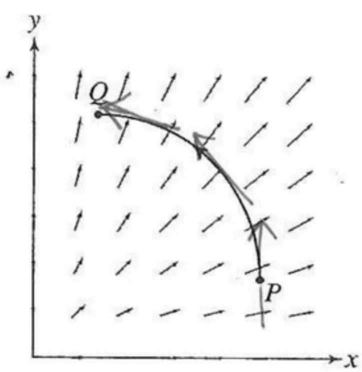
Note: the closed curves are oriented counterclockwise and the others are oriented $P \to Q$.



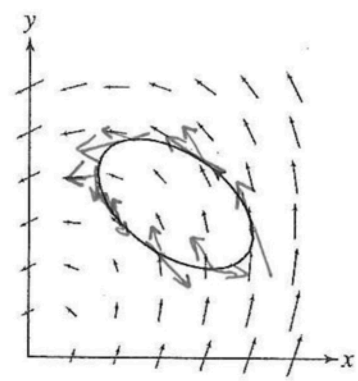
negative zero positive



negative zero positive



negative zero positive



negative zero positive

(d) $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ is zero if (circle one)

$\boldsymbol{F}$ is tangent to S at every point $\boldsymbol{F}$ is perpendicular to S at every point