20W-MATH32B-3 Final Exam

ZACK HIRSCHHORN

TOTAL POINTS

97 / 100

QUESTION 1

1 Q1 8 / 8

✓ + **2 pts** Parametrization

✓ + **2 pts** Derivative of parametrization and magnitude

✓ + **2 pts** Correct integral set up

✓ + **1 pts** Attempt at integration

✓ + **1 pts** Correct answer

+ **1 pts** Error in integral set up

+ **1 pts** Error in parametrization

+ **0 pts** No attempt

QUESTION 2

2 Q2 7 / 8

✓ + **4 pts** Correct bounds (no penalty if incorrect units)

✓ + **2 pts** Attempt at integration

+ **2 pts** Correct answer with correct units

+ **1 pts** Incorrect answer with correct units (some mistake in integration or evaluating at bounds)

✓ + **1 pts** Correct answer with incorrect units

+ **2 pts** Incorrect bounds or partially correct bounds

QUESTION 3

3 Q3 4 / 5

+ **3 pts** Correct bounds

✓ + **1 pts** Function in corresponding coordinate system

✓ + **1 pts** Integrating factor in corresponding coordinate system

✓ + **2 pts** At least one incorrect bound

+ **0 pts** No attempt

+ **2 pts** Did not leave as 3D integral, but correct 2D integral

QUESTION 4

4 Q4 5 / 5

✓ + **3 pts** Correct bounds

✓ + **1 pts** Function in corresponding coordinate system

✓ + **1 pts** Integrating factor in corresponding coordinate system

+ **2 pts** At least one bound incorrect

+ **0 pts** No attempt

QUESTION 5

5 Q5a 2 / 2

✓ - **0 pts** Correct

- **1 pts** Minor mistakes (including confusion of curl vs curl_z)

- **2 pts** Major mistakes, incorrect, etc

QUESTION 6

6 Q5b 6 / 6

✓ - **0 pts** Correct

- **1 pts** Minor errors (missing cos(1), etc)

- **3 pts** Correct param's / direction vectors, but integrating F1-F1+F2-F2

- **4 pts** Reasoned -C2=C4, etc, no parametrizations, no consideration of F

- **4 pts** Shows r but not r', claims each dot product is 0 with no justification (no consideration of F)

- **5 pts** No use of dot product, evaluation of integrals doesn't make sense

- **3 pts** Correct param's/direction vectors, but uses F(endpoint) rather than F(param)

- **4 pts** Used F(endpoint) to justify F along curve (no param)

- **4 pts** Incorrect notion of line integration, but recognized that both components of F have roots at 0,1 correctly

- **2 pts** Many minor errors

- **3 pts** Did half of the curves correctly
- **6 pts** Nothing
- **5 pts** Some ideas, but nothing helpful

QUESTION 7

7 Q6 **10 / 10**

✓ **- 0 pts** Correct
- **3 pts** Incorrect parameterisation
- **2 pts** Incorrect normal
- **3 pts** Incorrect surface integral formula.
- **2 pts** Incorrect final answer.

QUESTION 8

8 Q7 **10 / 10**

✓ **- 0 pts** Correct
- **2 pts** Did not complete loop.
- **3 pts** Incorrect form of Green's Theorem.
- **2 pts** Incorrect line integral (Not counting orientation, should be 1)
- **2 pts** Incorrect double integral (Should be 6xArea=3\\pi+6) .
- **1 pts** Incorrect final answer.

QUESTION 9

9 Q8 **5 / 6**

- **0 pts** Correct

✓ **- 1 pts Orientation error**
- **2 pts** Used unnormalized normal vector
- **2 pts** Parametrization mistake
- **5 pts** Used Green's Theorem without changing coordinates first
- **6 pts** No significant work

QUESTION 10

10 Q9 **6 / 6**

✓ **- 0 pts** Correct
- **2 pts** Generic arithmetic error
- **3 pts** Counted the total contribution of S_1, ..., S_5 as 5
- **2 pts** Orientation/sign error
- **6 pts** No significant progress

QUESTION 11

11 Q10 **8 / 8**

✓ **- 0 pts** Correct
- **2 pts** Integration error
- **7 pts** Tried to apply Stokes' Theorem
- **8 pts** No significant work

(1) Spherical, in this case

QUESTION 12

12 Q11 **6 / 6**

✓ **- 0 pts** Correct
- **1 pts** Minor mistakes / unclear logical flow
- **3 pts** Logic backwards in meaningful way
- **4 pts** Most pieces there, but combined with nonsense / no logical flow
- **4 pts** Attempted curl(f grad g) directly, did not use product rule properly
- **4 pts** Translated product rule and used Stokes, but nothing else helpful.
- **5 pts** Believed both sides were equal to zero, no use of product rule or Stokes
- **4 pts** Believed f curl(grad g) was a scalar function, thus could not be dotted with vector dS
- **5 pts** Wrote Stokes with a product, but didn't use product rule or get any further
- **6 pts** Nothing

QUESTION 13

13 Q12 **4 / 4**

✓ **- 0 pts** Correct
- **1 pts** Small errors
- **2 pts** Some good ideas but also unclear
- **3 pts** Did an example only
- **3 pts** Tried to relate to "bent volume" being empty because the vectors are tangent
- **3 pts** Algebraic mistakes with triple product, no mention of orthogonality
- **4 pts** Nothing helpful

QUESTION 14

14 Q13 **6 / 6**

✓ **- 0 pts** **Correct**

- 3 pts all signs wrong (completely backwards)

- 1 pts one sign wrong

- 2 pts two signs wrong, but still has some + some -

- 3 pts three signs wrong

- 4 pts No signs

- 6 pts Nothing helpful

QUESTION 15

15 Q14 **10 / 10**

✓ **- 0 pts** **All correct**

- 2 pts A wrong

- 2 pts B wrong

- 2 pts C wrong

- 2 pts D wrong

- 2 pts E wrong

1. [8 pts] Suppose we have a thin linear rod in $\mathbb{R}^3$ with endpoints $(1,1,0)$ and $(0,1,1)$, whose mass density is given by the function

$$\delta_M(x,y,z) = 1 + xyz \quad \text{grams per unit length.}$$

Find the total mass of the rod (in grams).

Total Mass $= \int_C \delta_M \, dr$

$r(t) = \langle 1-t, 1, t \rangle \quad 0 \le t \le 1$

$r'(t) = \langle -1, 0, 1 \rangle$

$|r'(t)| = \sqrt{2}$

$TM = \int_{t=0}^{1} (1 + (1-t)t) \cdot \sqrt{2} \, dt$

$TM = \sqrt{2} \int_{t=0}^{1} -t^2 + t + 1 \, dt$

$= \sqrt{2} \left[-\frac{1}{3}t^3 + \frac{1}{2}t^2 + t \right]_0^1$

$= \sqrt{2}\left(\frac{7}{6}\right) = \boxed{\frac{7\sqrt{2}}{6}}$

1 Q1 8 / 8

- ✓ + 2 pts Parametrization
- ✓ + 2 pts Derivative of parametrization and magnitude
- ✓ + 2 pts Correct integral set up
- ✓ + 1 pts Attempt at integration
- ✓ + 1 pts Correct answer
- + 1 pts Error in integral set up
- + 1 pts Error in parametrization
- + 0 pts No attempt

2. [8 pts] Suppose you are waiting for train A and your friend is waiting for train B at the station. Let X denote the wait time for train A, while Y denotes the wait time for train B. Both X and Y are in *minutes*. Suppose that the two wait times have a joint probability density function

$$p(x, y) = 12e^{-4x-3y}.$$

Suppose you are only willing to wait *one hour* for a train. What is the probability that you will board your train after your friend boards hers? That is to say, what is the probability that train A arrives after train B but before one hour has passed? YOU DO NOT NEED TO SIMPLIFY YOUR ANSWER IN THIS PROBLEM

$B < A < 1 \Longrightarrow Y < X < 1$ — lower bounds are 0 and not $-\infty$, because it doesn't make sense for minutes to be < 0.

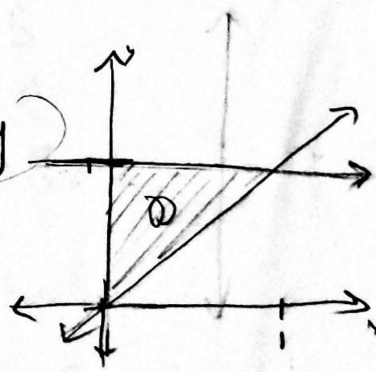
First, show that $\int_{x=0}^{\infty}\int_{y=0}^{\infty} p\, dA = 1$

$$\int_{x=0}^{\infty}\int_{y=0}^{\infty} 12e^{-4x-3y}\,dy\,dx = \int_{x=0}^{x=\infty}\left[-4e^{-4x-3y}\right]_0^{\infty} dx$$

$$= \int_{x=0}^{\infty} 4e^{-4x}\,dx = \left[-e^{-4x}\right]_0^{\infty} = 1$$

Second, to calculate the probability, calculate the integral using the proper bounds

$$\int_{y=0}^{1}\int_{x=y}^{x=1} 12e^{-4x-3y}\,dx\,dy = \int_{y=0}^{1}\left[-3e^{-4x-3y}\right]_y^1 dy$$



$$= \int_{y=0}^{1} -3e^{-4-3y} + 3e^{-7y}\,dy$$

$$= \left[e^{-4-3y} + \frac{3}{7}e^{-7y}\right]_0^1 = e^{-7} - \frac{3}{7}e^{-7} - e^{-4} + \frac{3}{7}$$

$$= \frac{4}{7}e^{-7} - e^{-4} + \frac{3}{7}$$

2 Q2 7 / 8

✓ + **4 pts** **Correct bounds (no penalty if incorrect units)**

✓ + **2 pts** **Attempt at integration**

+ **2 pts** Correct answer with correct units

+ **1 pts** Incorrect answer with correct units (some mistake in integration or evaluating at bounds)

✓ + **1 pts** **Correct answer with incorrect units**

+ **2 pts** Incorrect bounds or partially correct bounds

3. [5 pts] Set up the bounds for the integral $\iiint_W y\,dV$ (but do NOT compute) where $\mathcal{W}$ is the portion of the solid region between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ where $y \geq 0$. If you use cylindrical or spherical coordinates, be sure to write both the bounds AND the integrand in your chosen coordinate system.

USING CYLANDRICAL.

$r^2 < z < 4$

$0 < r < \sqrt{z}$

$0 < \theta < 2\pi$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=r^2}^{4} r^2 \sin\theta \, dz\, dr\, d\theta$$

4. [5 pts] Set up the bounds for the integral $\iiint_W z\,dV$ (but do NOT compute) where $\mathcal{W}$ is the region between the two spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 25$ in the first octant (x, y, z all positive). If you use cylindrical or spherical coordinates, be sure to write both the bounds AND the integrand in your chosen coordinate system.

SPHERICAL

$2 \leq \rho \leq 5$

$0 \leq \phi \leq \frac{\pi}{2}$

$0 \leq \theta \leq \frac{\pi}{2}$

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{2}} \int_{\rho=2}^{5} \rho^3 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta$$

3 Q3 **4 / 5**

- \+ **3 pts** Correct bounds
- ✓ + **1 pts** **Function in corresponding coordinate system**
- ✓ + **1 pts** **Integrating factor in corresponding coordinate system**
- ✓ + **2 pts** **At least one incorrect bound**
- \+ **0 pts** No attempt
- \+ **2 pts** Did not leave as 3D integral, but correct 2D integral

3. [5 pts] Set up the bounds for the integral $\iiint_W y\,dV$ (but do NOT compute) where $\mathcal{W}$ is the portion of the solid region between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ where $y \geq 0$. If you use cylindrical or spherical coordinates, be sure to write both the bounds AND the integrand in your chosen coordinate system.

USING CYLANDRICAL.

$z = r^2$
$z = 4$
$r\sin\theta \geq 0$

$r^2 < z < 4$

$0 < r < \sqrt{z}$

$0 < \theta < 2\pi$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2}^{4} r^2\sin\theta\,dz\,dr\,d\theta$$

4. [5 pts] Set up the bounds for the integral $\iiint_W z\,dV$ (but do NOT compute) where $\mathcal{W}$ is the region between the two spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=25$ in the first octant (x, y, z all positive). If you use cylindrical or spherical coordinates, be sure to write both the bounds AND the integrand in your chosen coordinate system.

SPHERICAL

$2 \leq \rho \leq 5$

$0 \leq \phi \leq \frac{\pi}{2}$

$0 \leq \theta \leq \frac{\pi}{2}$

$$\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\frac{\pi}{2}}\int_{\rho=2}^{5} \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

4 Q4 5 / 5

- ✓ + **3 pts** **Correct bounds**
- ✓ + **1 pts** **Function in corresponding coordinate system**
- ✓ + **1 pts** **Integrating factor in corresponding coordinate system**
- \+ **2 pts** At least one bound incorrect
- \+ **0 pts** No attempt

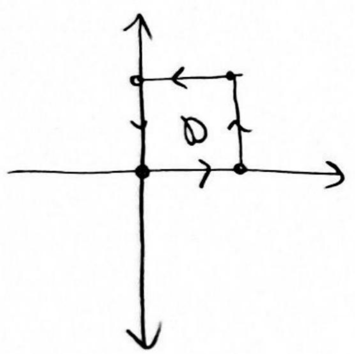
5. Let $\vec{\mathbf{F}}(x, y) = \langle y^2 - y, (x^3 - x)\cos(y^2)\rangle$, and let C be the unit square in the first quadrant with vertices $(0,0), (1,0), (1,1), (0,1)$ traversed counter-clockwise.

(a) [2 pts] Show that $\vec{\mathbf{F}}$ is *not* conservative.

$$\text{curl}(\vec{F}) = \langle 0, 0, 2y - 1 - 3x^2\cos(y^2) - \cos(y^2)\rangle \neq \vec{0}$$

Because $\text{curl}(\vec{F}) \neq \vec{0}$, $\vec{F}$ is not conservative

(b) [6 pts] Show that, despite the answer above, $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$. *Hints: Draw C, and think about how $\vec{\mathbf{F}}$ behaves along each side separately.*



Not using hint, because easier to prove using Green's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}_z(\vec{F})\, dA$$

$$= \int_{y=0}^{1} \int_{x=0}^{1} 2y - 1 - 3x^2\cos(y^2) - \cos(y^2)\, dx\, dy$$

$$= \int_{y=0}^{1} \left[2yx - x - x^3\cos(y^2) - \cos(y^2)\right]_0^1 dy$$

$$= \int_{y=0}^{1} 2y - 1\, dy = \left[y^2 - y\right]_0^1 = \boxed{0}$$

5 Q5a 2 / 2

- ✓ **- 0 pts** Correct
- **- 1 pts** Minor mistakes (including confusion of curl vs curl_z)
- **- 2 pts** Major mistakes, incorrect, etc

5. Let $\vec{\mathbf{F}}(x,y) = \langle y^2 - y, (x^3 - x)\cos(y^2)\rangle$, and let C be the unit square in the first quadrant with vertices $(0,0), (1,0), (1,1), (0,1)$ traversed counter-clockwise.

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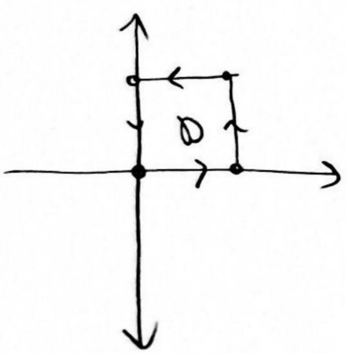

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$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}_z(\vec{F})\, dA$$

$$= \int_{y=0}^{1} \int_{x=0}^{1} 2y - 1 - 3x^2\cos(y^2) - \cos(y^2)\, dx\, dy$$

$$= \int_{y=0}^{1} \Big[2yx - x - x^3\cos(y^2) - \cos(y^2)\Big]_0^1 dy$$

$$= \int_{y=0}^{1} 2y - 1\, dy = \Big[y^2 - y\Big]_0^1 = \boxed{0}$$

6 Q5b **6 / 6**

✓ **- 0 pts** **Correct**

- 1 pts Minor errors (missing cos(1), etc)

- 3 pts Correct param's / direction vectors, but integrating F1-F1+F2-F2

- 4 pts Reasoned -C2=C4, etc, no parametrizations, no consideration of F

- 4 pts Shows r but not r', claims each dot product is 0 with no justification (no consideration of F)

- 5 pts No use of dot product, evaluation of integrals doesn't make sense

- 3 pts Correct param's/direction vectors, but uses F(endpoint) rather than F(param)

- 4 pts Used F(endpoint) to justify F along curve (no param)

- 4 pts Incorrect notion of line integration, but recognized that both components of F have roots at 0,1 correctly

- 2 pts Many minor errors

- 3 pts Did half of the curves correctly

- 6 pts Nothing

- 5 pts Some ideas, but nothing helpful

6. [10 pts] Evaluate $\iint_S y^2 z dS$ where S is the surface $x = \sqrt{3}y + \frac{\sqrt{5}}{2}z^2$ with $-1 \le y \le 1$ and $0 \le z \le 1$.

$u = y$, $v = z$

$G(u,v) = \langle \sqrt{3}u + \frac{\sqrt{5}}{2}v^2, u, v \rangle$ $\quad -1 \le u \le 1 \quad 0 \le v \le 1$

$T_u = \langle \sqrt{3}, 1, 0 \rangle$

$T_v = \langle \sqrt{5}v, 0, 1 \rangle$

$N = \langle 1, -\sqrt{3}, -\sqrt{5}v \rangle$

$\|N\| = \sqrt{1+3+5v^2} = \sqrt{4+5v^2}$

$$\iint_S y^2 z\, dS = \iint_D u^2 v \|N\|\, du\, dv$$

$$= \int_{u=-1}^{1}\int_{v=0}^{1} u^2 v (4+5v^2)^{\frac{1}{2}}\, dv\, du$$

$$= \int_{u=-1}^{1} \left[\frac{1}{15}u^2 (4+5v^2)^{\frac{3}{2}} \right]_0^1 du$$

$$= \int_{u=-1}^{1} \frac{1}{15}u^2 (27-8)\, du$$

$$= \frac{19}{15}\int_{u=-1}^{1} u^2\, du = \frac{19}{15}\left[\frac{1}{3}u^3\right]_{-1}^{1} = \frac{19}{15}\left(\frac{1}{3}+\frac{1}{3}\right)$$

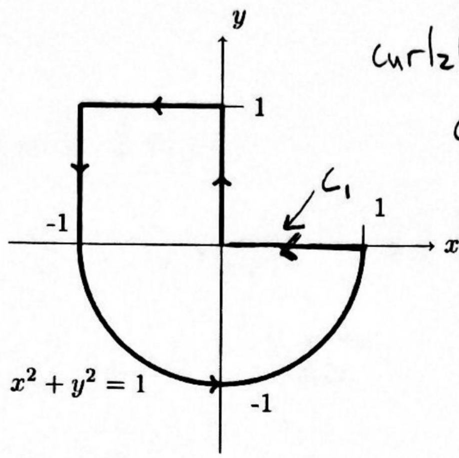
$$= \frac{38}{45}$$

7 Q6 10 / 10

✓ - **0 pts** **Correct**

- **3 pts** Incorrect parameterisation
- **2 pts** Incorrect normal
- **3 pts** Incorrect surface integral formula.
- **2 pts** Incorrect final answer.

7. [10 pts] Compute $\int_C \langle e^{\sin y}, xe^{\sin y}\cos y + 6x\rangle \cdot d\vec{r}$ where C is the curve shown below starting at $(0,0)$ and ending at $(1,0)$. *Hints: Green's Theorem*



$\text{curl}_z(\vec{F}) = \cos(y)e^{\sin(y)} - \cos(y)e^{\sin y} + 6$

$\text{curl}_z(\vec{F}) = 6$

According to Greens Theorem:

$$\int_C \vec{F}\cdot d\vec{r} + \int_{C_1} \vec{F}\cdot d\vec{r} = \iint_D \text{curl}_z(\vec{F})\, dA$$

$r(t) = \langle 1-t, 0\rangle \quad 0 \le t \le 1$

$r'(t) = \langle -1, 0\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D 6\, dA - \int_{t=0}^{1} \langle 1, 1-t + 6 - 6t\rangle \cdot \langle -1, 0\rangle\, dt$$

$$\int_C \vec{F}\cdot d\vec{r} = 6(\text{Area of } D) - \int_{t=0}^{1} -1\, dt$$

$$= 6\left(1 + \frac{\pi}{2}\right) - \left[-t\right]_0^1$$

$$= 6 + 3\pi + 1$$

$$\boxed{= 7 + 3\pi}$$

8 Q7 10 / 10

✓ **- 0 pts** Correct

- 2 pts Did not complete loop.

- 3 pts Incorrect form of Green's Theorem.

- 2 pts Incorrect line integral (Not counting orientation, should be 1)

- 2 pts Incorrect double integral (Should be 6xArea=3\\pi+6) .

- 1 pts Incorrect final answer.

8. **[6 pts]** Compute the line integral $\oint_C \langle y, -2z, 4x\rangle \cdot d\vec{\mathbf{r}}$ where C is a circle of radius 2 drawn on the plane $x + 2y + 3z = 4$ which is oriented *clockwise* when viewed from above looking downwards.
Hints: Stokes' Theorem

$\begin{matrix} y & -2z & 4x & y & -2z & 4x \\ d_x & d_y & d_z & d_x & d_y & d_z \end{matrix}$

Stokes Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = -\iint_S (\text{curl}(\vec{F})\cdot\vec{n})\,d\vec{S}$$

$$= -\iint_S \langle -2, 4, 1\rangle \cdot \tfrac{1}{\sqrt{14}}\langle 1, 2, 3\rangle\, d\vec{S}$$

$$= -\iint_S \tfrac{1}{\sqrt{14}}(-2 + 8 + 3)\, d\vec{S}$$

$$= -\tfrac{9}{\sqrt{14}}\iint_S d\vec{S}$$

$$= \tfrac{-9}{\sqrt{14}}(\text{Area of } S) = \tfrac{-9}{\sqrt{14}}\cdot 4\pi = \boxed{\tfrac{-36\pi}{\sqrt{14}}}$$

9. **[6 pts]** Let $\mathcal{W}$ a 3-dimensional solid cube, with each edge having length 2 (in other words, $\mathcal{W}$ is a solid box with length=width=height=2). As such, the boundary $\partial\mathcal{W}$ consists of six square faces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_6$, all of which are oriented outwards. Suppose further we have a vector field $\vec{\mathbf{F}}$ that satisfies the following facts:

- $\text{Div}\,\vec{\mathbf{F}} = 3$
- For the first five faces of the cube, the (outward) flux values are all equal:

$$\iint_{\mathcal{S}_1} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iint_{\mathcal{S}_2} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \cdots = \iint_{\mathcal{S}_5} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = 5.$$

Find the (outward) flux value $\iint_{\mathcal{S}_6} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}$ for the sixth and final face.

According to Divergence Theorem:

$$\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S} + \iint_{S_4} \vec{F}\cdot d\vec{S} + \iint_{S_5} \vec{F}\cdot d\vec{S} + \iint_{S_6} \vec{F}\cdot d\vec{S} = \iiint_W \text{div}(\vec{F})\,dV$$

$$\iint_{S_6} F\cdot d\vec{S} = \iiint 3\,dV - 25$$

$$= 3(\text{Volume of Cube}) - 25$$

$$= 3(8) - 25 = \boxed{-1}$$

9 Q8 5 / 6

- **- 0 pts** Correct
- ✓ **- 1 pts Orientation error**
- **- 2 pts** Used unnormalized normal vector
- **- 2 pts** Parametrization mistake
- **- 5 pts** Used Green's Theorem without changing coordinates first
- **- 6 pts** No significant work

8. [6 pts] Compute the line integral $\oint_C \langle y, -2z, 4x\rangle \cdot d\vec{\mathbf{r}}$ where C is a circle of radius 2 drawn on the plane $x + 2y + 3z = 4$ which is oriented *clockwise* when viewed from above looking downwards.
Hints: Stokes' Theorem

$y \quad -2z \quad 4x \quad y \quad -2z \quad 4x$
$d_x \quad d_y \quad d_z \quad d_x \quad d_y \quad d_z$

Stokes Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = -\iint_S (\text{curl}(\vec{F})\cdot\vec{n})\, d\vec{S}$$

$$= -\iint_S \langle -2, 4, 1\rangle \cdot \frac{1}{\sqrt{14}}\langle 1, 2, 3\rangle\, d\vec{S}$$

$$= -\iint_S \frac{1}{\sqrt{14}}(-2 + 8 + 3)\, d\vec{S}$$

$$= -\frac{9}{\sqrt{14}}\iint_S d\vec{S}$$

$$= \frac{-9}{\sqrt{14}}(\text{Area of } S) = \frac{-9}{\sqrt{14}}\cdot 4\pi = \boxed{\frac{-36\pi}{\sqrt{14}}}$$

9. [6 pts] Let $\mathcal{W}$ a 3-dimensional solid cube, with each edge having length 2 (in other words, $\mathcal{W}$ is a solid box with length=width=height=2). As such, the boundary $\partial\mathcal{W}$ consists of six square faces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_6$, all of which are oriented outwards. Suppose further we have a vector field $\vec{\mathbf{F}}$ that satisfies the following facts:

- $\text{Div}\,\vec{\mathbf{F}} = 3$
- For the first five faces of the cube, the (outward) flux values are all equal:

$$\iint_{\mathcal{S}_1} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iint_{\mathcal{S}_2} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \cdots = \iint_{\mathcal{S}_5} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = 5.$$

Find the (outward) flux value $\iint_{\mathcal{S}_6} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}$ for the sixth and final face.

According to Divergence Theorem;

$$\iint_{S_1}\vec{F}\cdot d\vec{S} + \iint_{S_2}\vec{F}\cdot d\vec{S} + \iint_{S_3}\vec{F}\cdot d\vec{S} + \iint_{S_4}\vec{F}\cdot d\vec{S} + \iint_{S_5}\vec{F}\cdot d\vec{S} + \iint_{S_6}\vec{F}\cdot d\vec{S} = \iiint_W \text{div}(\vec{F})\, dV$$

$$\iint_{S_6}\vec{F}\cdot d\vec{S} = \iiint 3\, dV - 25$$

$$= 3(\text{Volume of Cube}) - 25$$

$$= 3(8) - 25 \boxed{= -1}$$

10 Q9 6 / 6

✓ **- 0 pts** **Correct**

- 2 pts Generic arithmetic error

- 3 pts Counted the total contribution of S_1, ..., S_5 as 5

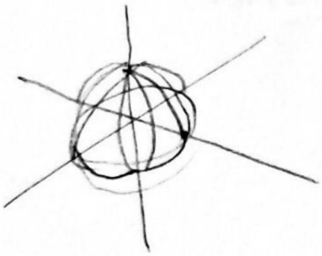
- 2 pts Orientation/sign error

- 6 pts No significant progress

10. [8 pts] Let S be the unit sphere in $\mathbb{R}^3$ with outward pointing normal. Show that

$$\iint_S \langle z^y, xy, e^{e^x - e^y}\rangle \cdot d\vec{\mathbf{S}} = 0.$$

Hints: We can view the unit sphere as the boundary of the solid unit ball in $\mathbb{R}^3$



Divergence Theorem

$$\oiint_S F \cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\, dV$$

$$= \iiint_W x\, dV$$

Polar!!!

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1} \rho^3 \sin^2\phi \cos\theta \, d\rho\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \left[\frac{1}{4}\rho^4 \sin^2\phi \cos\theta\right]_0^1 d\phi\, d\theta$$

$$= \frac{1}{4}\left(\int_{\theta=0}^{2\pi} \cos\theta\, d\theta \cdot \int_{\phi=0}^{\pi} \sin^2\phi\, d\phi\right)$$

for $\int_{x=0}^{\frac{2\pi}{2}} \sin^2 x = \frac{2\pi}{4}$

$$= \frac{1}{4}\left(\left[-\sin\theta\right]_0^{2\pi} \cdot \int_{\phi=0}^{\pi} \sin^2\phi\, d\phi\right)$$

$$= \frac{1}{4}\left(0 \cdot \int_{\phi=0}^{\pi} \sin^2\phi\, d\phi\right)$$

$$\boxed{= 0}$$

11 Q10 **8 / 8**

- ✓ **- 0 pts** Correct
- **- 2 pts** Integration error
- **- 7 pts** Tried to apply Stokes' Theorem
- **- 8 pts** No significant work

1 Spherical, in this case

11. [6 pts] It can be proved that, for any scalar function f and vector field $\vec{\mathbf{F}}$, the curl operator satisfies the following type of product rule:

$$\text{Curl}(f\vec{\mathbf{F}}) = f\text{Curl}\vec{\mathbf{F}} + \vec{\nabla}f \times \vec{\mathbf{F}}.$$

Taking this product rule as given (no need to prove it), use Stokes' Theorem to prove that

$$\iint_S (\vec{\nabla}f \times \vec{\nabla}g) \cdot d\vec{\mathbf{S}} = \oint_{\partial S} (f\vec{\nabla}g) \cdot d\vec{\mathbf{r}}$$

for any oriented surface $\mathcal{S}$.

$$\oint_{\partial s} (f\vec{\nabla}g) \cdot d\vec{r} = \iint_S \text{curl}(f\vec{\nabla}g) \cdot d\vec{S}$$

the curl of the gradient of something always $= 0$ so...

$$= \iint_S (f\,\text{curl}(\vec{\nabla}g) + \vec{\nabla}f \times \vec{\nabla g}) \cdot d\vec{S}$$

$$= \iint_S (\vec{\nabla}f \times \vec{\nabla}g) \cdot d\vec{S}$$

12. [4 pts] Suppose $\mathcal{S}$ is an oriented surface with a parametrization $G(u, v)$, so that the tangent vector fields $\vec{\mathbf{T}}_u$ and $\vec{\mathbf{T}}_v$ can be viewed as vector fields on the surface $\mathcal{S}$. Explain why we must always have

$$\iint_S \vec{\mathbf{T}}_u \cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{T}}_v \cdot d\vec{\mathbf{S}} = 0.$$

Because $\vec{T}_u$ and $\vec{T}_v$ are always perpendicular to $d\vec{S}$

12 Q11 6 / 6

✓ **- 0 pts** Correct

- 1 pts Minor mistakes / unclear logical flow

- 3 pts Logic backwards in meaningful way

- 4 pts Most pieces there, but combined with nonsense / no logical flow

- 4 pts Attempted curl(f grad g) directly, did not use product rule properly

- 4 pts Translated product rule and used Stokes, but nothing else helpful.

- 5 pts Believed both sides were equal to zero, no use of product rule or Stokes

- 4 pts Believed f curl(grad g) was a scalar function, thus could not be dotted with vector dS

- 5 pts Wrote Stokes with a product, but didn't use product rule or get any further

- 6 pts Nothing

11. [6 pts] It can be proved that, for any scalar function f and vector field $\vec{\mathbf{F}}$, the curl operator satisfies the following type of product rule:

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for any oriented surface S.

$$\oint_{\partial S} (f\vec{\nabla} g)\cdot d\vec{r} = \iint_S \text{curl}(f\vec{\nabla} g)\cdot d\vec{S}$$

the curl of the gradient of something always $=0$ so...

$$= \iint_S (f\,\text{curl}(\vec{\nabla} g) + \vec{\nabla} f \times \vec{\nabla} g)\cdot d\vec{S}$$

$$= \iint_S (\vec{\nabla} f \times \vec{\nabla} g)\cdot d\vec{S}$$

12. [4 pts] Suppose S is an oriented surface with a parametrization $G(u, v)$, so that the tangent vector fields $\vec{\mathbf{T}}_u$ and $\vec{\mathbf{T}}_v$ can be viewed as vector fields on the surface S. Explain why we must always have

$$\iint_S \vec{\mathbf{T}}_u \cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{T}}_v \cdot d\vec{\mathbf{S}} = 0.$$

Because $\vec{T}_u$ and $\vec{T}_v$ are always perpendicular to $d\vec{S}$

13 Q12 **4 / 4**

✓ **- 0 pts** **Correct**

- 1 pts Small errors

- 2 pts Some good ideas but also unclear

- 3 pts Did an example only

- 3 pts Tried to relate to "bent volume" being empty because the vectors are tangent

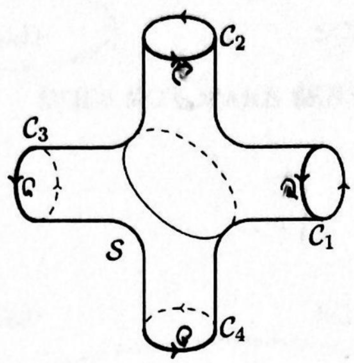
- 3 pts Algebraic mistakes with triple product, no mention of orthogonality

- 4 pts Nothing helpful

13. [6 pts] Consider the 'thickened plus sign' surface S shown below, oriented *outwards*. Suppose we are given the following circulation values for a vector field $\vec{\mathbf{F}}$ on the various boundary-circles:

$$\oint_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 1, \quad \oint_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \sqrt{2}, \quad \oint_{C_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \pi, \quad \oint_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = e.$$

What is the value of $\iint_S \operatorname{Curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$?



Be careful with orientations! On the top and bottom of the plus, the boundary-circles are oriented counter-clockwise when viewed from above looking downward; on the left and right of the plus, the boundary-circles are oriented counter-clockwise when viewed from the right looking leftward. There is a depth curve drawn on the surface to aid in visualization.

$$\iint_S \operatorname{Curl}(\vec{F}) \cdot d\vec{S} = -\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r}$$

$$= -1 - \sqrt{2} + \pi + e$$

14 Q13 **6 / 6**

✓ **- 0 pts** **Correct**

- 3 pts all signs wrong (completely backwards)

- 1 pts one sign wrong

- 2 pts two signs wrong, but still has some + some -

- 3 pts three signs wrong

- 4 pts No signs

- 6 pts Nothing helpful

14. For each of these questions, $\vec{\mathbf{F}}$ denotes a vector field while f denotes a scalar function. Indicate whether the operations described result in vector fields, scalar functions, or do not make sense. Circle your answers (or write down clearly on separate paper). No justification needed.

(a) [2 pts] $\mathrm{Curl}(\mathrm{Curl}(\vec{\mathbf{F}})) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE

(b) [2 pts] $\mathrm{Div}(\mathrm{Div}(\vec{\mathbf{F}})) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE

(c) [2 pts] $\mathrm{Div}(\vec{\nabla}(f)) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE

(d) [2 pts] $\vec{\nabla}(\mathrm{Curl}(\vec{\mathbf{F}})) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE

(e) [2 pts] $\vec{\nabla}(\mathrm{Div}(\vec{\mathbf{F}})) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE

15 Q14 10 / 10

✓ - **0 pts** **All correct**

- **2 pts** A wrong

- **2 pts** B wrong

- **2 pts** C wrong

- **2 pts** D wrong

- **2 pts** E wrong