

STUDENT NAME: _____

STUDENT ID NUMBER: _____

DISCUSSION SECTION NUMBER: _____

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

| Page | Points | Score |
|--------|--------|-------|
| 2 | 8 | |
| 3 | 8 | |
| 4 | 10 | |
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| 7 | 10 | |
| 8 | 12 | |
| 9 | 8 | |
| 10 | 10 | |
| 11 | 6 | |
| 12 | 10 | |
| Total: | 100 | |

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1. [8 pts] Suppose we have a thin linear rod in \mathbb{R}^3 with endpoints $(1, 1, 0)$ and $(0, 1, 1)$, whose mass density is given by the function

$$\delta_M(x, y, z) = 1 + xyz \quad \text{grams per unit length.}$$

Find the total mass of the rod (in grams).

2. [8 pts] Suppose you are waiting for train A and your friend is waiting for train B at the station. Let X denote the wait time for train A , while Y denotes the wait time for train B . Both X and Y are in *minutes*. Suppose that the two wait times have a joint probability density function

$$p(x, y) = 12e^{-4x-3y}.$$

Suppose you are only willing to wait *one hour* for a train. What is the probability that you will board your train after your friend boards hers? That is to say, what is the probability that train A arrives after train B but before one hour has passed? **YOU DO NOT NEED TO SIMPLIFY YOUR ANSWER IN THIS PROBLEM**

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3. [5 pts] Set up the bounds for the integral $\iiint_{\mathcal{W}} y dV$ (but do NOT compute) where \mathcal{W} is the portion of the solid region between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ where $y \geq 0$. If you use cylindrical or spherical coordinates, be sure to write both the bounds AND the integrand in your chosen coordinate system.
4. [5 pts] Set up the bounds for the integral $\iiint_{\mathcal{W}} z dV$ (but do NOT compute) where \mathcal{W} is the region between the two spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 25$ in the first octant (x, y, z all positive). If you use cylindrical or spherical coordinates, be sure to write both the bounds AND the integrand in your chosen coordinate system.

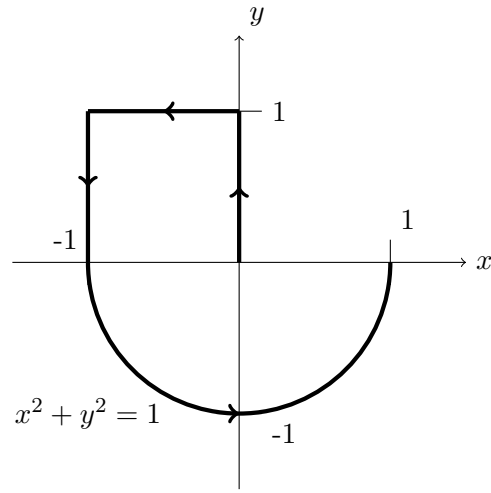
5. Let $\vec{\mathbf{F}}(x, y) = \langle y^2 - y, (x^3 - x) \cos(y^2) \rangle$, and let \mathcal{C} be the unit square in the first quadrant with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$ traversed counter-clockwise.

(a) [2 pts] Show that $\vec{\mathbf{F}}$ is *not* conservative.

(b) [6 pts] Show that, despite the answer above, $\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$. *Hints: Draw \mathcal{C} , and think about how $\vec{\mathbf{F}}$ behaves along each side separately.*

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6. [10 pts] Evaluate $\iint_{\mathcal{S}} y^2 z dS$ where \mathcal{S} is the surface $x = \sqrt{3}y + \frac{\sqrt{5}}{2}z^2$ with $-1 \leq y \leq 1$ and $0 \leq z \leq 1$.

7. [10 pts] Compute $\int_{\mathcal{C}} \langle e^{\sin y}, xe^{\sin y} \cos y + 6x \rangle \cdot d\vec{\mathbf{r}}$ where \mathcal{C} is the curve shown below starting at $(0,0)$ and ending at $(1,0)$. *Hints: Green's Theorem*



8. [6 pts] Compute the line integral $\oint_{\mathcal{C}} \langle y, -2z, 4x \rangle \cdot d\vec{\mathbf{r}}$ where \mathcal{C} is a circle of radius 2 drawn on the plane $x + 2y + 3z = 4$ which is oriented *clockwise* when viewed from above looking downwards.
Hints: Stokes' Theorem

9. [6 pts] Let \mathcal{W} a 3-dimensional solid cube, with each edge having length 2 (in other words, \mathcal{W} is a solid box with length=width=height=2). As such, the boundary $\partial\mathcal{W}$ consists of six square faces $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_6$, all of which are oriented outwards. Suppose further we have a vector field $\vec{\mathbf{F}}$ that satisfies the following facts:

- $\text{Div } \vec{\mathbf{F}} = 3$
- For the first five faces of the cube, the (outward) flux values are all equal:

$$\iint_{\mathcal{S}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{\mathcal{S}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \dots = \iint_{\mathcal{S}_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 5.$$

Find the (outward) flux value $\iint_{\mathcal{S}_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ for the sixth and final face.

10. [8 pts] Let \mathcal{S} be the unit sphere in \mathbb{R}^3 with outward pointing normal. Show that

$$\iint_{\mathcal{S}} \langle z^y, xy, e^{e^x - e^y} \rangle \cdot d\vec{\mathbf{S}} = 0.$$

Hints: We can view the unit sphere as the boundary of the solid unit ball in \mathbb{R}^3

11. [6 pts] It can be proved that, for any scalar function f and vector field $\vec{\mathbf{F}}$, the curl operator satisfies the following type of product rule:

$$\text{Curl}(f\vec{\mathbf{F}}) = f\text{Curl}\vec{\mathbf{F}} + \vec{\nabla}f \times \vec{\mathbf{F}}.$$

Taking this product rule as given (no need to prove it), use Stokes' Theorem to prove that

$$\iint_{\mathcal{S}} (\vec{\nabla}f \times \vec{\nabla}g) \cdot d\vec{\mathbf{S}} = \oint_{\partial\mathcal{S}} (f\vec{\nabla}g) \cdot d\vec{\mathbf{r}}$$

for any oriented surface \mathcal{S} .

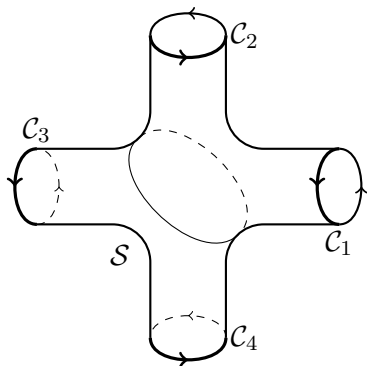
12. [4 pts] Suppose \mathcal{S} is an oriented surface with a parametrization $G(u, v)$, so that the tangent vector fields $\vec{\mathbf{T}}_u$ and $\vec{\mathbf{T}}_v$ can be viewed as vector fields on the surface \mathcal{S} . Explain why we must always have

$$\iint_{\mathcal{S}} \vec{\mathbf{T}}_u \cdot d\vec{\mathbf{S}} = \iint_{\mathcal{S}} \vec{\mathbf{T}}_v \cdot d\vec{\mathbf{S}} = 0.$$

13. [6 pts] Consider the ‘thickened plus sign’ surface \mathcal{S} shown below, oriented *outwards*. Suppose we are given the following circulation values for a vector field $\vec{\mathbf{F}}$ on the various boundary-circles:

$$\oint_{\mathcal{C}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 1, \quad \oint_{\mathcal{C}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \sqrt{2}, \quad \oint_{\mathcal{C}_3} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \pi, \quad \oint_{\mathcal{C}_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = e.$$

What is the value of $\iint_{\mathcal{S}} \text{Curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$?



Be careful with orientations! On the top and bottom of the plus, the boundary-circles are oriented counter-clockwise when viewed from above looking downward; on the left and right of the plus, the boundary-circles are oriented counter-clockwise when viewed from the right looking leftward. There is a depth curve drawn on the surface to aid in visualization.

14. For each of these questions, $\vec{\mathbf{F}}$ denotes a vector field while f denotes a scalar function. Indicate whether the operations described result in vector fields, scalar functions, or do not make sense. Circle your answers (or write down clearly on separate paper). No justification needed.

(a) [2 pts] $\text{Curl}(\text{Curl}(\vec{\mathbf{F}})) =$

VECTOR FIELD

SCALAR FUNCTION

DOES NOT MAKE SENSE

(b) [2 pts] $\text{Div}(\text{Div}(\vec{\mathbf{F}})) =$

VECTOR FIELD

SCALAR FUNCTION

DOES NOT MAKE SENSE

(c) [2 pts] $\text{Div}(\vec{\nabla}(f)) =$

VECTOR FIELD

SCALAR FUNCTION

DOES NOT MAKE SENSE

(d) [2 pts] $\vec{\nabla}(\text{Curl}(\vec{\mathbf{F}})) =$

VECTOR FIELD

SCALAR FUNCTION

DOES NOT MAKE SENSE

(e) [2 pts] $\vec{\nabla}(\text{Div}(\vec{\mathbf{F}})) =$

VECTOR FIELD

SCALAR FUNCTION

DOES NOT MAKE SENSE

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