16.1 - 16.4

Math 32B
Calculus of Several Variables

Midterm 1

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 46 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ____________

Student ____________

Discussic ____________

Question	Points	Score
1	9	7
2	15	12
3	12	12
4	10	10
Total:	46	41

19

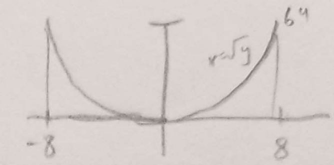
22

Problem 1.

7 Calculate the following 2D integrals using whatever methods you like.

(a) [4pts.] $\int_{x=-8}^{8}\int_{y=0}^{x^2} 4x(x^2+1)y^3\,dy\,dx$.

(b) [5pts.] $\int_{y=0}^{\pi/4}\int_{x=2y}^{\pi/2} \frac{\sin x}{x}\,dx\,dy$.



2 a. $\int_{-8}^{8}\left[(4x^3+4x)\left(\frac{1}{4}y^4\right)\right]_0^{x^2} dx$

$= \int_{-8}^{8}\left[(x^3+x)((x^2)^4)\right] dx = \int_{-8}^{8} (x^3+x)(x^8)\,dx$

symmetry ✗ $= 2\int_0^8 (x^{11}+x^9)\,dx = 2\left[\frac{1}{12}x^{12}+\frac{1}{10}x^{10}\right]_0^8$

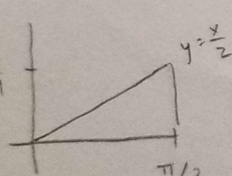
$= 2\left(\frac{1}{12}(8)^{12}+\frac{1}{10}(8)^{10}\right)$

$= \boxed{\frac{1}{6}(8)^{12}+\frac{1}{5}(8)^{10}}$

b. $0 \le y \le \frac{\pi}{4} \qquad 2y \le x \le \frac{\pi}{2}$

$y = \frac{x}{2} \rightarrow$ if $y = \frac{\pi}{4}$, $x = \frac{\pi}{2}$

$\rightarrow$ if $y=0$, $x=0$



$\int_{x=0}^{\pi/2}\int_{y=0}^{x/2} \frac{\sin x}{x}\,dy\,dx$

$= \int_0^{\pi/2}\left[y\frac{\sin x}{x}\right]_0^{x/2} dx = \int_0^{\pi/2}\left[\frac{x}{2}\,\frac{\sin x}{x}\right] dx$

$= \int_0^{\pi/2}\frac{1}{2}\sin x\,dx = \frac{1}{2}\int_0^{\pi/2}\sin x\,dx$

$= -\frac{1}{2}\left[\cos x\right]_0^{\pi/2} = -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right)-\cos(0)\right]$

$= -\frac{1}{2}\left[0-(1)\right] = \boxed{\frac{1}{2}}$ ✓

Problem 2.

For each of the following multiple choice questions, you do NOT need to show any scratch work, and we will NOT grade your scratch work.

You will receive 0 points for an incorrect response, 1 point for a blank response, and 3 points for a correct response. Indicate your responses by CIRCLING.

The first two parts concern 2D integrals; the last two parts concern 3D integrals.

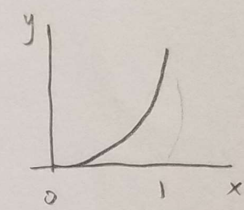
For the last part, the rubric applies twice since there are two correct responses.

if $x = \frac{1}{2}$, $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

if $x = 2$, $\sin(\pi) = 0$

(a) [3pts.] If D is the region in the first quadrant between $y = x^2$ and $y = \sin(\frac{\pi x}{2})$, then D has the description:

1. $0 \le x \le \pi$, $0 \le y \le \sin(\frac{\pi x}{2})$
2. $0 \le x \le \pi$, $\sin(\frac{\pi x}{2}) \le y \le x^2$
3. $0 \le x \le 1$, $\sin(\frac{\pi x}{2}) \le y \le x^2$
4. $0 \le x \le 1$, $x^2 \le y \le \sin(\frac{\pi x}{2})$



$x^2 = \sin(\frac{\pi x}{2})$

@ $x = 0$

@ $x = 1$

(b) [3pts.] The integral of $2\sqrt{x^2+y^2}$ over the disk $x^2+y^2 \le 1$ is:

1. $\frac{2\pi}{3}$
2. π
3. $\frac{4\pi}{3}$
4. 2π

$r^2 \le 1 \Rightarrow r \le 1$

$\int_0^{2\pi}\int_0^1 2r^2\,dr\,d\theta = \int_0^{2\pi}\left[\frac{2}{3}r^3\right]_0^1 d\theta$

$= \int_0^{2\pi}\frac{2}{3}\,d\theta = \frac{2}{3}(\theta)_0^{2\pi}$

$= \frac{2}{3}(2\pi) = \frac{4\pi}{3}$

(c) [3pts.] If $B = [-\frac{1}{2}, \frac{1}{2}] \times [-1, 3] \times [0, 1]$, then $\int\int\int_B 2y\ dV$ is equal to:

1. -4
2. 0
3. 4
4. 8

$\int_{-1/2}^{1/2}\int_0^1\int_{-1}^3 2y\ dy\,dz\,dx$

$\int_{-1/2}^{1/2}\int_0^1 [y^2]_{-1}^3\,dz\,dx = \int_{-1/2}^{1/2}[8z]_0^1\,dx$

$= \int_{-1/2}^{1/2} 8\,dx = 8x\Big]_{-1/2}^{1/2}$

$= 8(\frac{1}{2}+\frac{1}{2}) = 8$

(d) [6pts.] $\int_{x=0}^{4}\int_{y=0}^{3x}\int_{z=0}^{8-2x} f(x,y,z)\ dz\ dy\ dx$

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can be rewritten as which TWO of the following integrals?

$0 \le z \le 8-2x$

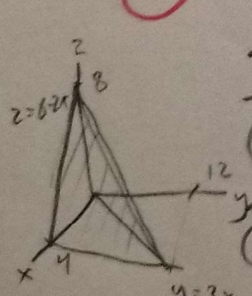
$0 \le y \le 3x$

$0 \le x \le 4$

$y = 3x \Rightarrow \frac{y}{3} = x$

$z = 8-2x = 8 - 2(\frac{y}{3})$

1. $\int_{y=0}^{3x}\int_{z=0}^{8-2x}\int_{x=0}^{4} f(x,y,z)\ dx\ dz\ dy$
2. $\int_{y=0}^{12}\int_{z=0}^{8-2y/3}\int_{x=0}^{4} f(x,y,z)\ dx\ dz\ dy$
3. $\int_{y=0}^{12}\int_{z=0}^{8-2y/3}\int_{x=y/3}^{4} f(x,y,z)\ dx\ dz\ dy$
4. $\int_{z=0}^{8}\int_{y=0}^{12-3z/2}\int_{x=y/3}^{4} f(x,y,z)\ dx\ dy\ dz$
5. $\int_{y=0}^{12}\int_{x=y/3}^{4}\int_{z=0}^{8-2x} f(x,y,z)\ dz\ dx\ dy$
6. $\int_{z=0}^{8}\int_{x=0}^{4-z/2}\int_{y=0}^{3x} f(x,y,z)\ dy\ dx\ dz$



$y = 3x$
$x = \frac{y}{3}$

$z = 8 - 2x$

$z \to 8 = -2x$

$\frac{8-z}{2} = x \Rightarrow 4 - \frac{1}{2}z = x$

Problem 3.

Let $\mathcal{E}$ be the region:

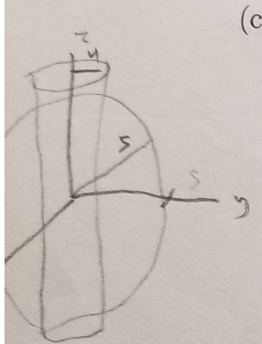
- inside the sphere $x^2 + y^2 + z^2 = 25$, that is, with $x^2 + y^2 + z^2 \leq 25$;
- outside the cylinder $x^2 + y^2 = 16$; that is, with $x^2 + y^2 \geq 16$.

The volume of $\mathcal{E}$ is given by the 3D integral $\int\int\int_{\mathcal{E}} 1\, dV$.

(a) [4pts.] Set up the integral in cylindrical polar coordinates in the order $dz\ dr\ d\theta$.

(b) [4pts.] Set up the integral in cylindrical polar coordinates in the order $dr\ dz\ d\theta$.

(c) [4pts.] Calculate the volume of $\mathcal{E}$ in any way that you like.



$r^2 + z^2 \leq 25$ $\qquad$ $r^2 \geq 16 \Rightarrow r \geq 4$

$r^2 \leq -z^2 + 25$ $\qquad$ $16 + z^2 \leq 25$

or $z^2 \leq 25 - r^2$ $\qquad$ $z^2 = 9$

$z = \pm\sqrt{25 - r^2}$ $\qquad$ $z = \pm 3$

a. $$\int_{\theta=0}^{2\pi} \int_{r=4}^{5} \int_{z=-\sqrt{25-r^2}}^{\sqrt{25-r^2}} 1\, r\, dz\, dr\, d\theta$$

b. $$\int_{\theta=0}^{2\pi} \int_{z=-3}^{3} \int_{r=4}^{\sqrt{25-z^2}} 1\, r\, dr\, dz\, d\theta$$

c. $$\int_{\theta=0}^{2\pi} \int_{4}^{5} \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} 1\, r\, dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{4}^{5} [zr]_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}}\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_4^5 \left(\sqrt{25-r^2} + \sqrt{25-r^2}\right) r\, dr\, d\theta$$

$$= 2\int_0^{2\pi} \int_4^5 r\sqrt{25-r^2}\, dr\, d\theta$$

$u = 25 - r^2$

$du = -2r\, dr$

$u = 25 - 16 = 9$

$u = 25 - 25 = 0$

$$= -\int_0^{2\pi} \int_9^0 \sqrt{u}\, du\, d\theta$$

$$= -\int_0^{2\pi} \left[\frac{2}{3} u^{3/2}\right]_9^0 d\theta$$

$$= -\int_0^{2\pi} -\frac{2}{3}(9)^{3/2}\, d\theta = \frac{2}{3}\int_0^{2\pi} 27\, d\theta = 18\,[\theta]_0^{2\pi} = \boxed{36\pi}$$

cone

$\rho\cos\phi = \rho\sin\phi$

satisfied when

$\rho = 0,$

$\phi = \frac{\pi}{4},$

sphere

$\rho^2 = 4\rho\cos\phi$

$0 \le \rho \le 4\cos\phi$

$\cos\phi = 0$ @ $\phi = \frac{\pi}{2}$

Problem 4.

Let $\mathcal{E}$ be the region:

- inside the sphere $x^2 + y^2 + (z-2)^2 = 4$, that is, with $x^2 + y^2 + (z-2)^2 \le 4$;
- above the cone $z = \sqrt{x^2+y^2}$, that is, with $z \ge \sqrt{x^2+y^2}$,
- in the third xy-quadrant, that is, with $x, y \le 0$.

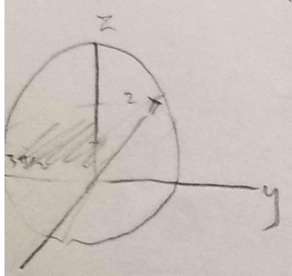
(a) [6pts.] Prepare to calculate the following integral in spherical coordinates:

$$\iiint_{\mathcal{E}} \frac{x}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\, dV.$$

$\cancel{\frac{\rho\sin\phi\cos\theta}{\rho\sin\phi(\rho)}}$

$= \frac{\cos\theta}{\rho}(\rho^2\sin\phi)$

$= \rho\sin\phi\cos\theta$



Noting that the sphere also has the equation $x^2 + y^2 + z^2 = 4z$ might be helpful. Although drawing a sketch is not required for full credit, if you are struggling to set up the integral, appropriate sketches might help with partial credit.

If you're a clever-clogs and think about such things, you do not need to worry about the undefinedness of the function along the z-axis.

(b) [4pts.] Show that

$$\iiint_{\mathcal{E}} \frac{x}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\, dV = \frac{8}{3}\left(\frac{1}{2\sqrt{2}} - 1\right).$$

You can use any method that you like.

a. $\displaystyle\int_{\theta=\pi}^{3\pi/2} \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{4\cos\phi} \rho\sin\phi\cos\theta \, d\rho\, d\phi\, d\theta$ ✓

b. $\displaystyle\frac{1}{2}\int_{\theta=\pi}^{3\pi/2} \int_{\phi=\frac{\pi}{4}}^{\pi/2} \left[\rho^2 \sin\phi\cos\theta\right]_0^{4\cos\phi} d\phi\, d\theta$

$\displaystyle= \frac{1}{2}\int_{\pi}^{3\pi/2}\int_{\pi/4}^{\pi/2} 4\cos^2\phi\sin\phi\cos\theta \, d\phi\, d\theta$

$\displaystyle= 2\int_{\pi}^{3\pi/2} \cos\theta \int_{\pi/4}^{\pi/2} \cos^2\phi\sin\phi \, d\phi\, d\theta$

$u = \cos\phi$

$du = -\sin\phi\, d\phi$

$u = \cos\left(\frac{\pi}{2}\right) = 0$

$u = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\displaystyle= -2\int_{\pi}^{3\pi/2} \cos\theta \int_{\sqrt{2}/2}^{0} u^2 \, du\, d\theta$

$\displaystyle= -2\int_{\pi}^{3\pi/2} \cos\theta \left[\frac{1}{3}u^3\right]_{\sqrt{2}/2}^{0} d\theta$

$\displaystyle= \frac{-2}{3}\int_{\pi}^{3\pi/2} \cos\theta\left(-\frac{2\sqrt{2}}{8}\right) d\theta = \frac{8}{3}\left(\frac{1}{2\sqrt{2}}\right)\int_{\pi}^{3\pi/2} \cos\theta\, d\theta$

$\displaystyle= \frac{6}{3}\left(\frac{1}{2\sqrt{2}}\right)(\sin\theta)_{\pi}^{3\pi/2} = \boxed{\frac{8}{3}\left(\frac{1}{2\sqrt{2}} - 1\right)}$ ✓