Exercise 1 (25 points)

(a) [10 pts.] Let $\mathcal{D}$ be the diamond-shaped region in $\mathbb{R}^2$ with vertices at $(0,0)$, (π,π), $(0,2\pi)$, $(-\pi,\pi)$. Find a map which transforms the region $\mathcal{D}_0 = [0,2\pi]\times[0,2\pi]$ in the (u,v)-plane into $\mathcal{D}$. Make a picture of both $\mathcal{D}_0$ and $\mathcal{D}$.

(b) [15 pts.] Use your answer from part (a) to evaluate the following integral:

$$\int\int_{\mathcal{D}} (x-y)^2 \sin^2(x+y)\, dA.$$

Hint: the following formula might come in handy: $\sin^2 x = \frac{1-\cos(2x)}{2}$.

(a) we know,

$(0,0) \to \Phi(0,0)$

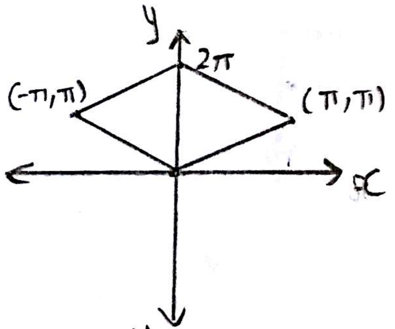
therefore:

$v=0 \to x=y$

$u=0 \to x=-y$

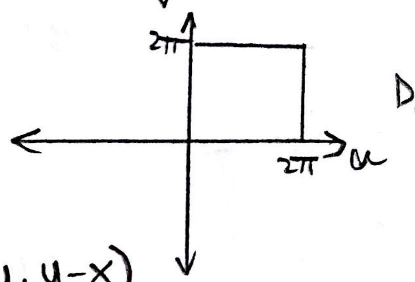
$u=2\pi \to x+y=2\pi$

$\therefore x+y=u$ — ①



$v=2\pi \to y-x=2\pi$

$\therefore y-x=v$ — ②



$\therefore x+y=u$
$y-x=v$

$\therefore G(x,y)=(x+y,\, y-x)$

(b) $Jac(G)=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ $\Big\}$ $\begin{bmatrix}\frac{du}{dx} & \frac{du}{dy}\\ \frac{dv}{dx} & \frac{dv}{dy}\end{bmatrix}$

$= 2$

$\therefore$ ~~Inverse JAC(G⁻¹) = 1/2~~

Jac(G^{-1}) is what we want

$$\int_0^{2\pi}\int_0^{2\pi} (-v)^2 \sin^2(u) |Jac(G)| du\, dv$$

$$1/2 \Rightarrow 2\int_0^{2\pi}\int_0^{2\pi} v^2\sin^2(u)\, du\, dv$$

$$\Rightarrow 2\left[\frac{v^3}{3}\right]_0^{2\pi} \int_0^{2\pi} \frac{1-\cos 2u}{2}\, du$$

-1

$$\Rightarrow \frac{16\pi^3}{3}\left[\frac{1}{2}u - \frac{\sin 2u}{4}\right]_0^{2\pi} \Rightarrow \frac{16\pi^3}{3}[\pi] \Rightarrow \frac{\cancel{16}^{\,4}}{3}\pi^4$$

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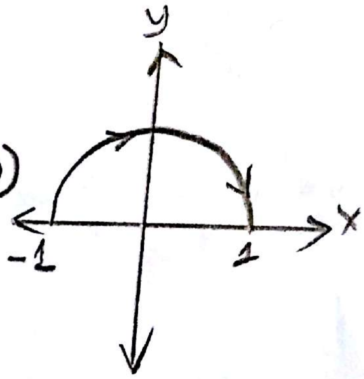
Exercise 2 (24 points)

(a) [12 pts.] Find the total charge on the upper semicircle $x^2+y^2=1$, $y\geq 0$, oriented clockwise, with charge density $\delta(x,y)=xy^3$;
(b) [12 pts.] Find the flux of the vector field $\mathbf{F}=\left(\frac{y^3}{[(x+2)^4+y^4]^{1/2}}, \frac{(x+2)^3}{[(x+2)^4+y^4]^{1/2}}\right)$ across the segment $1\leq x\leq 3$ oriented left to rigth.



(a) $r(t) = (\cos(t), \sin(t))$

[take inverse later] where path goes from $(-1,0)$ to $(-1,0)$

$\therefore \quad 0 \leq t \leq \pi$

$\therefore \quad r'(t) = (-\sin(t), \cos(t))$

$\|r'(t)\| = 1$

$\delta(r(t)) = \cos(t)\sin^3(t)$

$\therefore$ charge $= -\int_0^\pi \cos(t)\sin^3(t)\,dt$

let $\sin(t) = u$

$\therefore \cos(t)\,dt = du$

charge $= -\int_0^\pi u^3\,du = \frac{u^4}{4}$

$= \frac{\sin(t)^4}{4}\Big|_0^\pi$

$= \underline{\underline{0}}$

(b) $F = \left(\frac{y^3}{((x+2)^4+y^4)^{1/2}}, \frac{(x+2)^3}{((x+2)^4+y^4)^{1/2}}\right)$

$r(t) = (t, 0)$, $1 \leq t \leq 3$

$r'(t) = (1, 0)$ $\quad \therefore$ $\underline{\underline{N(t) = (0,-1)}}$

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$$F(r(t)) = \left(0, \frac{(t+2)^3}{(t+2)^2}\right) = (0, t+2)$$

$\therefore$ flux $= \int_1^3 -t-2 \, dt$

$$= \left[-\frac{t^2}{2} - 2t\right]_1^3$$

$$= \left[-\frac{9}{2} - 6\right] - \left[-\frac{1}{2} - 2\right]$$

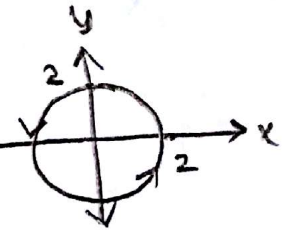
$$= -\frac{21}{2} + \frac{5}{2} = -16/2 = \underline{\underline{-8}}$$

Exercise 3 (26 points)

(a) [10 pts.] Let $\mathbf{F}(x,y) = \left(\frac{-y+x}{x^2+y^2}, \frac{x+y}{x^2+y^2}\right)$ be a planar vector field. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the circle $x^2+y^2=4$ oriented counterclockwise.
(b) [8 pts.] Is $\mathbf{F}$ conservative on $\mathcal{D} = \{(x,y) \neq (0,0)\}$? Explain.
(c) [4 pts.] Show that $\mathbf{F}$ satisfies the cross-partial condition.
(d) [4 pts.] Show that $\mathbf{F}$ is conservative on $\mathcal{D} = \{(x,y) | x > 0\}$

(a) $F(x,y) = \left(\frac{-y+x}{x^2+y^2}, \frac{x+y}{x^2+y^2}\right)$

parameterization: $r(t) = (2\cos(t), 2\sin(t))$
$0 \le t \le 2\pi$



$\cancel{\int F}$ $F(r(t)) = \left[\frac{2\cos(t) - 2\sin(t)}{4}, \frac{2\cos(t) + 2\sin(t)}{4}\right]$

$r'(t) = (-2\sin(t), 2\cos(t))$

$\int_C F\cdot dr = \int_0^{2\pi} F(r(t))\cdot r'(t)\, dt$

$= \int_0^{2\pi} \frac{-4\sin(t)\cos(t) + 4\sin^2(t)}{4} + \frac{4\cos^2(t) + 4\sin(t)\cos(t)}{4}\, dt$

$= \int_0^{2\pi} \cancel{-\sin(t)\cos(t) +} \sin^2(t) + \cos^2(t) \cancel{+ \sin(t)\cos(t)}$

$= \int_0^{2\pi} 1\, dt = \underline{\underline{2\pi}}$ OK

(b) There are multiple reasons for F being non-conservative. Firstly, there is a hole in the domain at $x=y=0$ which means that D is not simply correct. Also, in the previous subpart, we get the $\int_C F\cdot dr$ over a complete path where $P=Q$. However, $\int F\cdot dr \neq 0$ which means the field is not conservative. $[f(P)-f(Q) \neq 0]$ OK!!
CORRECT!!

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(c) condition: $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$

$$\Rightarrow \frac{\partial\left[\frac{-y+x}{x^2+y^2}\right]}{\partial y} = \frac{\partial\left[\frac{x+y}{x^2+y^2}\right]}{\partial x}$$

using product rule

$$\Rightarrow \frac{\partial}{\partial y} = (-y+x)(x^2+y^2)^{-1}$$

$$= -1(x^2+y^2)^{-1} + (-y+x)\cdot -2y\,(x^2+y^2)^{-2}$$

$$= \frac{-1}{x^2+y^2} - \frac{2y(x-y)}{(x^2+y^2)^2} = \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2} = \underline{\underline{\frac{y^2-2xy-x^2}{(x^2+y^2)^2}}}$$

$$\frac{\partial}{\partial x} = (x+y)(x^2+y^2)^{-1}$$

$$= (x^2+y^2)^{-1} + (x+y)\cdot -2x(x^2+y^2)^{-2}$$

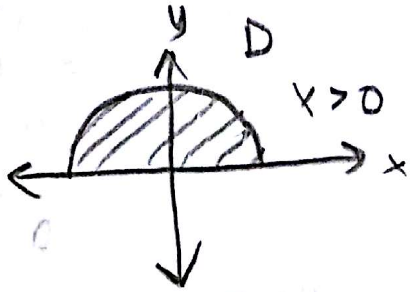
$$= \frac{1}{x^2+y^2} - \frac{2x(x+y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2-2xy}{(x^2+y^2)^2} = \underline{\frac{y^2-2xy-x^2}{(x^2+y^2)}}$$

∴ the F satisfies the condition

ok

(d) curl(F) = 0 as proven in part (c)

if $x > 0$, the domain is simple without holes



$$F(x,y) = \left(\frac{-y+x}{r^2}, \frac{x+y}{r^2}\right)$$

~~$\int F_x\,dx$~~ ⇒ these two conditions prove that $F(x,y)$ is conservative

ok!

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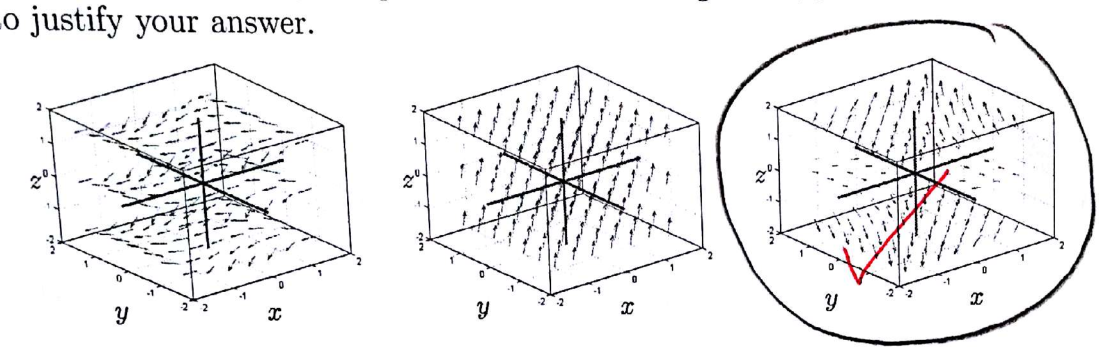
Exercise 4 (25 points)

(a) [5 pts.] Given the three-dimensional vector field $\mathbf{F}(x, y, z) = (\frac{y}{1+x^2}, \tan^{-1} x, 2z)$ which of the following is a plot of $\mathbf{F}$? Circle the right one, you do not need to justify your answer.



(b) [5 pts.] Compute div($\mathbf{F}$) and curl($\mathbf{F}$);

(c) [15 pts.] Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ over the unit circle in the (x, y)-plane clockwise oriented.

(b) $F(x,y,z) = \left(\frac{y}{1+x^2}, \tan^{-1}(x), 2z\right)$ -3

$$\text{div}(F) = \left(\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial y}, \frac{\partial F_3}{\partial z}\right) = \left(\frac{-2xy}{(1+x^2)^2}, \quad , 2\right)$$

1

$$\text{curl}(F) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} \frac{y}{1+x^2} \\ \tan^{-1}(x) \\ 2z \end{bmatrix} = \left[\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(\tan^{-1} x)\right] i$$

$$- \left[\frac{\partial}{\partial x}(2z) - \frac{\partial}{\partial z}\left(\frac{y}{1+x^2}\right)\right] j$$

$$+ \left[\frac{\partial}{\partial x}[\tan^{-1}(x)] - \frac{\partial}{\partial y}\left[\frac{y}{1+x^2}\right]\right] k$$

$$= [0-0]i - [0-0]j + \left[\frac{1}{1+x^2} - \frac{1}{1+x^2}\right]k$$

$$= \underline{\underline{0}}$$

(c) $\int \frac{y}{1+x^2}\, dx = y \tan^{-1}(x) + f(y,z)$

$\int \tan^{-1}(x)\, dy = y \tan^{-1}(x) + g(x,z)$

$\int 2z\, dz = z^2 + h(x,y)$

$\therefore f(x,y,z) = y \tan^{-1}(x) + z^2$ — potential function

Since curl(F) = 0, there is a potential function in a simple area, F(x,y,z) is a conservative vector field

$\therefore \int_C F \cdot dr = \oint F \cdot dr = \underline{0}$