MATH 32B-1, Fall 2016, MIDTERM I
Calculus of Several Variables

Name: ....

UID number:...

Section & TA: ..

DO NOT START UNTIL TOLD TO DO SO

You have 50 minutes to complete the exam. There are 4 problems, worth a total of 100 points.

No books, calculators, or notes of any kind are allowed.

Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

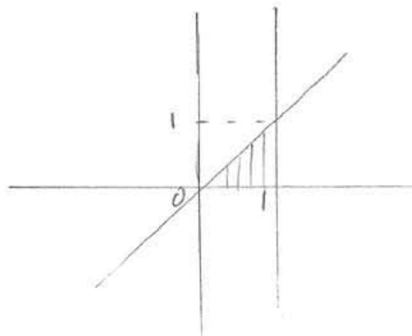
Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

1	20
2	24
3	21
4	25
Total	90

Exercise 1 (20 points)

(a) [6pts.] Draw the plane region in the first quadrant bounded by the $x-$axis, the line $x = 1$ and the line $y = x$.
(b) [14pts.] Compute the double integral of the function $f(x, y) = \frac{xy}{1+x^4}$ over this region.

a)



b) $D: 0 \leq x \leq 1 \qquad 0 \leq y \leq x$

$$\int_0^1 \int_0^x \frac{xy}{1+x^4}\, dy\, dx$$
$$= \int_0^1 \frac{x}{1+x^4} \left(\frac{1}{2}\right) y^2 \Big|_0^x dx$$
$$= \frac{1}{2} \int_0^1 \frac{x}{1+x^4} (x^2)\, dx$$
$$= \frac{1}{2} \int_0^1 \frac{x^3}{1+x^4}\, dx$$
$$= \frac{1}{2} \left(\frac{1}{4}\right) \int_1^2 \frac{1}{u}\, du$$
$$= \frac{1}{8} \ln|u| \Big|_1^2$$
$$= \frac{1}{8}(\ln 2 - \ln 1)$$
$$= \boxed{\frac{1}{8} \ln 2}$$

$u = 1 + x^4 \qquad x = 0 \quad u = 1$
$du = 4x^3 dx \qquad x = 1 \quad u = 2$
$\frac{1}{4} du = x^3 dx$

20

Exercise 2 (25 points)

Evaluate the following integrals using the method you like.

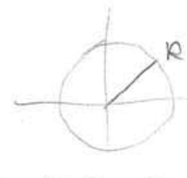
(a) [10pts.] $\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{1+x^2+y^2}\, dA$;

(b) [15pts.] $\int_{x=0}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{2} xz \; dz\, dy\, dx$

a) $\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$

$0 \le \theta \le 2\pi$

$0 \le r \le R$



$= \int_0^{2\pi} \int_0^{R} \frac{r}{1+r^2}\, dr\, d\theta$

$= \int_0^{2\pi} d\theta \int_0^{R} \frac{r}{r^2+1}\, dr$

$u = r^2+1$

$du = 2r\, dr$

$\frac{1}{2} du = r\, dr$

$r=0 \quad u=1$

$r=R \quad u=R^2+1$

$= 2\pi\left(\frac{1}{2}\right) \int_1^{R^2+1} \frac{1}{u}\, du$

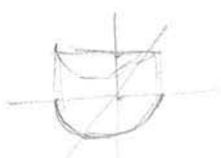
$= \boxed{\pi \ln |R^2+1|}$

b) $\int_{x=0}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{2} xz \; dz\,dy\,dx$

$0 \le z \le 2$

$0 \le \theta \le \pi$

$0 \le r \le 1$



$= \int_0^{\pi} \int_0^{1} \int_0^{2} r\cos(\theta) z \cdot r\, dz\, dr\, d\theta$

$\pi/2$

$= \int_0^{\pi} \cos\theta\, d\theta \int_0^1 r^2 dr \int_0^2 z\, dz$

$\pi/2$

$= \sin\theta \Big|_0^{\pi} \cdot \frac{1}{3} r^3 \Big|_0^1 \cdot \frac{1}{2} z^2 \Big|_0^2$

$= 0 \cdot \frac{1}{3} \cdot 2$

$= \boxed{0}$

24

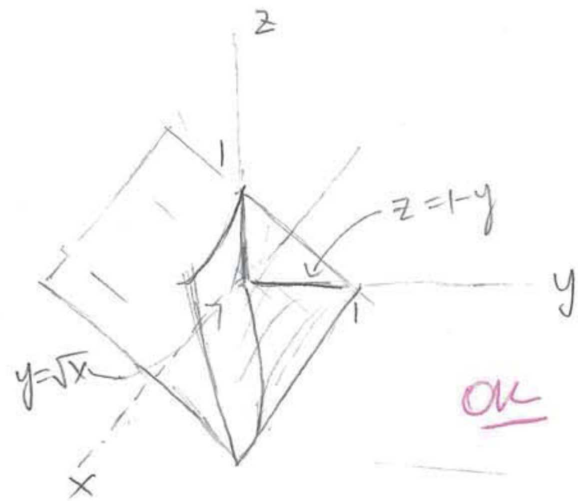
Exercise 3 (25 points)

(a) [10pts.] Draw the region $\mathcal{W}$ bounded by the yz-plane, xy-plane, the plane $z = 1 - y$, and $y = \sqrt{x}$.
(b) [15pts.] Write down (but do not evaluate) three different triple integrals of that compute the volume of this region by projecting $\mathcal{W}$ onto the
(*i*) $xy-$plane (*ii*) $yz-$plane (*iii*) $xz-$plane.

a)

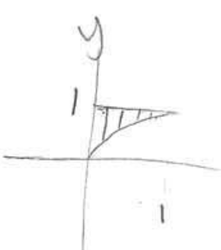
$z = 1-y \qquad y = 1-z$

$y = \sqrt{x}$



OK

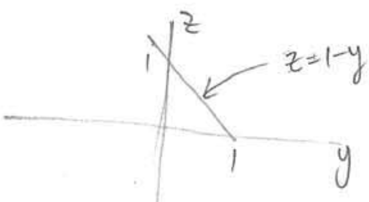
b) xy-plane $\qquad 0 \le z \le 1-y \qquad 0 \le y \le 1$



$$\boxed{\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dz\, dy\, dx}$$

OK

yz-plane



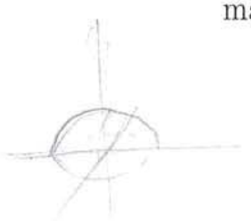
$$\boxed{\int_0^1 \int_0^{1-y} \int_0^{y^2} dx\, dz\, dy}$$

OK

xz-plane

Exercise 4 (25 points)

Find the total mass and the z-coordinate of the center of mass of the solid bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 0$, assuming a mass density $\delta(x, y, z) = z$.



$$z_{cm} = \frac{M_{xy}}{M} \qquad z = \rho\cos\phi$$

$0 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{2}$

$$M = \iiint_W \delta(x,y,z)\, dV$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 \rho\cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \cos\phi \sin\phi\, d\phi \cdot \int_0^2 \rho^3 d\rho$$
$$= 2\pi \cdot \int_0^1 u\, du \cdot \frac{1}{4}\rho^4 \Big|_0^2$$
$$= 2\pi \cdot \frac{1}{2}u^2 \Big|_0^1 \cdot 4$$
$$= \boxed{4\pi}$$

$u = \sin\phi$

$du = \cos\phi\, d\phi$

$\phi = 0 \quad u = 0$

$\phi = \frac{\pi}{2} \quad u = 1$

$$M_{xy} = \iiint_W z\, \delta(x,y,z)\, dV$$
$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2\cos^2\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^2\phi \sin\phi\, d\phi \int_0^2 \rho^4 d\rho$$
$$= 2\pi \cdot \int_1^0 -u^2 du \cdot \frac{1}{5}\rho^5 \Big|_0^2$$
$$= \frac{2\pi\, 32}{5}\left(\frac{1}{3}\right)u^3 \Big|_0^1$$
$$= \frac{64\pi}{15}$$

$u = \cos\phi$

$du = -\sin\phi\, d\phi$

$\phi = 0$

$u = 1$

$\phi = \frac{\pi}{2}$

$u = 0$

$$z_{cm} = \frac{M_{xy}}{M} = \frac{64\pi}{15} \cdot \frac{1}{4\pi} = \boxed{\frac{16\pi}{15}}$$