Quiz 4 Answer

1. $xy + 2yz + 3xz$.

2(i) $G(y,z) = (y^2 + z^2, y, z)$ for y, z in $D = \{(y,z) |\, y^2 + z^2 \le 1\}$.

2(ii) $\dfrac{\partial G}{\partial y} = (2y, 1, 0)$ and $\dfrac{\partial G}{\partial z} = (2z, 0, 1)$. So

$$N(y,z) = \frac{\partial G}{\partial y} \times \frac{\partial G}{\partial z} = \begin{bmatrix} 2y \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 2z \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2y \\ -2z \end{bmatrix}.$$

$G(y,z) = \begin{bmatrix} 0.05 \\ 0.1 \\ 0.2 \end{bmatrix}$ when $y = 0.1$ and $z = 0.2$. We have $N(0.1, 0.2) = \begin{bmatrix} 1 \\ -0.2 \\ -0.4 \end{bmatrix}$.

The tangent plane of $\mathcal{S}$ at $\begin{bmatrix} 0.05 \\ 0.1 \\ 0.2 \end{bmatrix}$ is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} |\, (x - 0.05) - 0.2(y - 0.1) - 0.4(z - 0.2) = 0 \right\}$$
$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} |\, x - 0.2y - 0.4z = -0.05 \right\}.$$

2(iii)

$$\int_{\mathcal{S}} xy^2 \, d(x,y,z)$$
$$= \int_D (y^2 + z^2) y^2 \|N(y,z)\| \, d(y,z)$$
$$= \int_D (y^2 + z^2) y^2 \sqrt{1 + 4y^2 + 4z^2} \, d(y,z)$$
$$= \int_0^{2\pi} \int_0^1 r^2 (r\cos\theta)^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta \quad \text{or} \quad \int_{-1}^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} (y^2 + z^2) y^2 \sqrt{1 + 4y^2 + 4z^2} \, dy \, dz$$
$$= 1.041619.$$

Alternative solution for question 2: next page

Alternative solution

2(i) $G(u,\theta) = (u^2, u\cos\theta, u\sin\theta)$ for y, z in $D = \{(u,\theta)|\, 0 \le u \le 1,\ 0 \le \theta \le 2\pi\}$.

2(ii) $\dfrac{\partial G}{\partial u} = (2u, \cos\theta, \sin\theta)$ and $\dfrac{\partial G}{\partial \theta} = (0, -u\sin\theta, u\cos\theta)$. So

$$N(u,\theta) = \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial \theta} = \begin{bmatrix} 2u \\ \cos\theta \\ \sin\theta \end{bmatrix} \times \begin{bmatrix} 0 \\ -u\sin\theta \\ u\cos\theta \end{bmatrix} = \begin{bmatrix} u \\ -2u^2\cos\theta \\ -2u^2\sin\theta \end{bmatrix}.$$

$G(u,\theta) = \begin{bmatrix} 0.05 \\ 0.1 \\ 0.2 \end{bmatrix}$ when $u = \sqrt{0.05}$ and $\theta = 1.107149$.

We have $N(\sqrt{0.05}, 1.107149) = \begin{bmatrix} \sqrt{0.05} \\ -0.044721 \\ -0.089442 \end{bmatrix}$.

The tangent plane of $\mathcal{S}$ at $\begin{bmatrix} 0.05 \\ 0.1 \\ 0.2 \end{bmatrix}$ is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid \sqrt{0.05}(x-0.05) - 0.044721(y-0.1) - 0.089442(z-0.2) = 0 \right\}$$
$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x - 0.2y - 0.4z = -0.05 \right\}.$$

2(iii)

$$\begin{aligned} &\int_{\mathcal{S}} xy^2 \, d(x,y,z) \\ &= \int_D u^2 (u\cos\theta)^2 \|N(u,\theta)\| \, d(u,\theta) \\ &= \int_D u^4 \cos^2\theta (u\sqrt{1+4u^2}) \, d(u,\theta) = \int_0^{2\pi} \int_0^1 u^5 \cos^2\theta \sqrt{1+4u^2} \, du = 1.041619. \end{aligned}$$

Remark: For 2(i), the domain could also be $-1 \le u \le 1$, $0 \le \theta \le \pi$. But it is incorrect for the domain to be $-1 \le u \le 1$, $0 \le \theta \le 2\pi$, because for each $x = u^2$, there are two possible u and with these two possible u, there is duplication when we allow both positive and negative radius, at the same time, in the polar coordinates $y = u\cos\theta$ and $z = u\sin\theta$.

On the other hand, for page 13 in Section 14, for each z, there is only one possible radius. Even though the radius z in the polar coordinates could be positive or negative, we are not allowing both positive and negative radius at the same time.