Quiz 5 Answer

1. Take the parametrization

$$G(z,t) = (z + \cos t, \sin t, z),$$

for (z,t) in $D = \{(z,t) |\, 0 \leq z \leq 1,\ 0 \leq t \leq 2\pi\}$. (To come up with this, note that if we fix z, then $(x - z, y)$ can be parametrized as $(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. To get a parametrization for (x, y, z), we need to write x, y, z in terms of two variables. We have
$x = z + \cos t,\ y = \sin t,\ z = z$.)

We have

$$\frac{\partial G}{\partial z} = (1, 0, 1) \quad \text{and} \quad \frac{\partial G}{\partial t} = (-\sin t, \cos t, 0),$$

so

$$N(z,t) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos t \\ -\sin t \\ \cos t \end{bmatrix}.$$

Since $N(z,t) \cdot \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} = -1 < 0$, the parametrization G is negatively oriented.

The flux of F across $\mathcal{S}$ is

$$\begin{aligned} &- \int_D F(G(z,t)) \cdot N(z,t)\, d(z,t) \\ &= - \int_0^{2\pi} \int_0^1 \begin{bmatrix} z \\ 2\sin t \\ z + \cos t \end{bmatrix} \cdot \begin{bmatrix} -\cos t \\ -\sin t \\ \cos t \end{bmatrix} dz\, dt \\ &= - \int_0^{2\pi} \int_0^1 -z\cos t - 2\sin^2 t + (z + \cos t)\cos t\, dz\, dt = \pi. \end{aligned}$$

Note: If we use the parametrization $G(t,z) = (z + \cos t, \sin t, z)$ then $N(z,t)$ becomes $\begin{bmatrix} \cos t \\ \sin t \\ -\cos t \end{bmatrix}$ and G is positively oriented. In the integral set up for the flux, there is no minus with the integral but the final answer is still π.

Alternative solution: Split the surface into

$$\mathcal{S}_1 = \{(x,y,z)|\,(x-z)^2+y^2=1,\ 0\le z\le 1,\ x-z\ge 0\}$$
$$\mathcal{S}_2 = \{(x,y,z)|\,(x-z)^2+y^2=1,\ 0\le z\le 1,\ x-z< 0\}.$$

$\mathcal{S}_1$: Take the parametrization

$$G_1(y,z) = (z+\sqrt{1-y^2}, y, z),$$

for (y,z) in $D=\{(y,z)|-1\le y\le 1,\ 0\le z\le 1\}$. We have

$$\frac{\partial G_1}{\partial y} = \left(-\frac{y}{\sqrt{1-y^2}}, 1, 0\right) \quad \text{and} \quad \frac{\partial G_1}{\partial z} = (1,0,1),$$

so

$$N_1(y,z) = \begin{bmatrix} 1 \\ y/\sqrt{1-y^2} \\ -1 \end{bmatrix}.$$

Since $N_1(y,z)\cdot\begin{bmatrix} \sqrt{1-y^2} \\ y \\ 0 \end{bmatrix} = \sqrt{1-y^2}+\dfrac{y^2}{\sqrt{1-y^2}} > 0$, the parametrization G is positively oriented.

The flux of F across $\mathcal{S}_1$ is

$$\begin{aligned}
&\int_D F(G_1(z,t))\cdot N_1(z,t)\,d(z,t)\\
&= \int_{-1}^{1}\int_0^1 \begin{bmatrix} z \\ 2y \\ z+\sqrt{1-y^2} \end{bmatrix}\cdot\begin{bmatrix} 1 \\ y/\sqrt{1-y^2} \\ -1 \end{bmatrix}\,dz\,dy\\
&= \int_{-1}^{1}\int_0^1 z+\frac{2y^2}{\sqrt{1-y^2}}-(z+\sqrt{1-y^2})\,dz\,dy = 1.570796.
\end{aligned}$$

$\mathcal{S}_2$: Take the parametrization

$$G_2(y,z) = (z-\sqrt{1-y^2}, y, z),$$

for (y,z) in $D=\{(y,z)|-1\le y\le 1,\ 0\le z\le 1\}$. We have

$$\frac{\partial G_2}{\partial y} = \left(\frac{y}{\sqrt{1-y^2}}, 1, 0\right) \quad \text{and} \quad \frac{\partial G_2}{\partial z} = (1,0,1),$$

so

$$N_2(y,z) = \begin{bmatrix} 1 \\ -y/\sqrt{1-y^2} \\ -1 \end{bmatrix}.$$

Since $N_2(y,z)\cdot\begin{bmatrix} -\sqrt{1-y^2} \\ y \\ 0 \end{bmatrix} = -\sqrt{1-y^2}-\dfrac{y^2}{\sqrt{1-y^2}} < 0$, the parametrization G is negatively oriented.

The flux of F across $\mathcal{S}_2$ is

$$
\begin{aligned}
&-\int_D F(G_2(z,t)) \cdot N_2(z,t)\, d(z,t) \\
&= -\int_{-1}^{1}\int_0^1 \begin{bmatrix} z \\ 2y \\ z-\sqrt{1-y^2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -y/\sqrt{1-y^2} \\ -1 \end{bmatrix} dz\, dy \\
&= -\int_{-1}^{1}\int_0^1 z - \frac{2y^2}{\sqrt{1-y^2}} - (z-\sqrt{1-y^2})\, dz\, dy = 1.570796.
\end{aligned}
$$

The flux of F across $\mathcal{S}$ is $1.570796 + 1.570796 = 3.141592$.

2. By Green's theorem,

$$
\begin{aligned}
&\int_{\mathcal{C}} y^3\, dx + x^2 y\, dx \\
&= -\int_D \frac{\partial x^2 y}{\partial x} - \frac{\partial y^3}{\partial y}\, d(x,y) \\
&= -\int_D 2xy - 3y^2\, d(x,y) = -\int_0^1 \int_{x^2}^{x+1} 2xy - 3y^2\, dy\, dx = 2.357143.
\end{aligned}
$$