Quiz 3 Answer

1(i) We have

$$\begin{aligned} W &= \{(x,y,z)|\, x^2+y^2+z^2 \le 3,\ y > x > 0,\ z > 0\} \\ &= \{(r\cos\theta, r\sin\theta, z)|\, r^2+z^2 \le 3,\ \sin\theta > \cos\theta > 0,\ r > 0,\ z > 0\} \\ &= \{(r\cos\theta, r\sin\theta, z)|\, 0 < z \le \sqrt{3-r^2},\ \frac{\pi}{4} < \theta < \frac{\pi}{2},\ r > 0\}. \end{aligned}$$

$$\begin{aligned} \int_W z\, d(x,y,z) &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\infty} \int_{[0,\sqrt{3-r^2}]} zr\, dz\, dr\, d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{3}} \int_0^{\sqrt{3-r^2}} zr\, dz\, dr\, d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{3}} \frac{1}{2}(3-r^2)r\, dr\, d\theta = \int_0^{\frac{\pi}{4}} \frac{9}{4} - \frac{9}{8}\, d\theta = \frac{9\pi}{32}. \end{aligned}$$

1(ii) We have

$$\begin{aligned} & W \\ =& \{(x,y,z)|\, x^2+y^2+z^2 \le 3,\ y > x > 0,\ z > 0\} \\ =& \{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)| \\ & 0 < \rho \le \sqrt{3},\ \rho\sin\phi\sin\theta > \rho\sin\phi\cos\theta > 0,\ \rho\cos\phi > 0,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\} \\ =& \{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)| \\ & 0 < \rho \le \sqrt{3},\ \sin\theta > \cos\theta > 0,\ \cos\phi > 0,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\} \\ =& \{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)|\, 0 < \rho \le \sqrt{3},\ 0 \le \phi < \frac{\pi}{2},\ \frac{\pi}{4} < \theta < \frac{\pi}{2}\}. \end{aligned}$$

$$\begin{aligned} \int_W z\, d(x,y,z) &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} (\rho\cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{9}{4} \cos\phi \sin\phi\, d\phi\, d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{9}{8} \sin(2\phi)\, d\phi\, d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{9}{8}\, d\theta = \frac{9\pi}{32}. \end{aligned}$$

Question 2 next page

2(i) We have

$$\begin{aligned} W &= \{(x,y,z)|\, x^2+y^2+z^2 \le 3,\ y>0,\ z\ge 1\} \\ &= \{(r\cos\theta, r\sin\theta, z)|\, r^2+z^2\le 3,\ \sin\theta>0,\ r>0,\ z\ge 1\} \\ &= \{(r\cos\theta, r\sin\theta, z)|\, 1\le z\le \sqrt{3-r^2},\ 0<\theta<\pi\}. \end{aligned}$$

$$\begin{aligned} \int_W z\, d(x,y,z) &= \int_0^\pi \int_0^\infty \int_{[1,\sqrt{3-r^2}]} zr\, dz\, dr\, d\theta \\ &= \int_0^\pi \int_0^{\sqrt{2}} \int_1^{\sqrt{3-r^2}} zr\, dz\, dr\, d\theta \\ &= \int_0^\pi \int_0^{\sqrt{2}} \frac{1}{2}(2-r^2) r\, dr\, d\theta = \int_0^\pi 1 - \frac{1}{2}\, d\theta = \frac{\pi}{2}. \end{aligned}$$

2(ii) We have

$$\begin{aligned} &W \\ =&\{(x,y,z)|\, x^2+y^2+z^2\le 3,\ y>0,\ z\ge 1\} \\ =&\{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)| \\ &0<\rho\le\sqrt{3},\ \rho\sin\phi\sin\theta>0,\ \rho\cos\phi\ge 1,\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi\} \\ =&\{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)| \\ &0<\rho\le\sqrt{3},\ \sin\theta>0,\ \rho\cos\phi\ge 1,\ \cos\phi>0,\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi\} \\ =&\{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)|\, 0<\rho\le\sqrt{3},\ \rho\cos\phi\ge 1,\ 0<\phi<\frac{\pi}{2},\ 0<\theta<\pi\} \\ =&\{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)|\ \sec\phi\le\rho\le\sqrt{3},\ 0<\phi<\frac{\pi}{2},\ 0<\theta<\pi\}. \end{aligned}$$

$$\begin{aligned} \int_W \frac{1}{\sqrt{x^2+y^2}}\, d(x,y,z) &= \int_0^\pi \int_0^{\frac{\pi}{2}} \int_{[\sec\phi,\sqrt{3}]} \frac{1}{\rho\sin\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^\pi \int_0^{\frac{\pi}{2}} \int_{[\sec\phi,\sqrt{3}]} \rho\, d\rho\, d\phi\, d\theta \\ &= \int_0^\pi \int_0^{0.955317} \int_{\sec\phi}^{\sqrt{3}} \rho\, d\rho\, d\phi\, d\theta \\ &= \int_0^\pi \int_0^{0.955317} \frac{1}{2}(3-\sec^2\phi)\, d\phi\, d\theta \\ &= \int_0^\pi 1.432976 - 0.707107\, d\theta = 2.280385. \end{aligned}$$