Midterm 2 Answer

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1.

$$\begin{aligned} D &= \{(r\cos\theta, r\sin\theta)|\, r\sin\theta > 0,\ \frac{1}{2} < r^2 < r\cos\theta,\ r > 0,\ 0 \le \theta \le 2\pi\} \\ &= \{(r\cos\theta, r\sin\theta)|\, \sin\theta > 0,\ r > \frac{1}{\sqrt{2}},\ r < \cos\theta,\ 0 \le \theta \le 2\pi\} \\ &= \{(r\cos\theta, r\sin\theta)|\, \sin\theta > 0,\ \cos\theta > 0,\ \frac{1}{\sqrt{2}} < r < \cos\theta,\ 0 \le \theta \le 2\pi\} \\ &= \{(r\cos\theta, r\sin\theta)|\, \frac{1}{\sqrt{2}} < r < \cos\theta,\ 0 < \theta < \frac{\pi}{2}\}. \end{aligned}$$

So

$$\int_D 1\, d(x,y) = \int_0^{\frac{\pi}{2}} \int_{[\frac{1}{\sqrt{2}},\cos\theta]} r\, dr\, d\theta = \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\sqrt{2}}}^{\cos\theta} r\, dr\, d\theta,$$

since for $0 \le \theta \le \frac{\pi}{2}$, we have $\frac{1}{\sqrt{2}} \le \cos\theta \Leftrightarrow \theta \le \frac{\pi}{4}$. So $\int_D 1\, d(x,y) = \frac{1}{8}$.

2.

$$\begin{aligned} &W \\ =&\{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)|\, \sqrt{2} < \rho < \sqrt{3},\ \rho\sin\phi > 2\rho\cos\phi > 0,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\} \\ =&\{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)|\, \sqrt{2} < \rho < \sqrt{3},\ \sin\phi > 2\cos\phi,\ 0 < \phi < \frac{\pi}{2},\ 0 \le \theta \le 2\pi\} \\ =&\{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)|\, \sqrt{2} < \rho < \sqrt{3},\ \tan\phi > 2,\ 0 < \phi < \frac{\pi}{2},\ 0 \le \theta \le 2\pi\} \\ =&\{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)|\, \sqrt{2} < \rho < \sqrt{3},\ \phi > 1.107149,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\}. \end{aligned}$$

So

$$\int_W x^2 + y^2 + z^2\, d(x,y,z) = \int_0^{2\pi} \int_{1.107149}^{\frac{\pi}{2}} \int_{\sqrt{2}}^{\sqrt{3}} \rho^2(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta = 5.581411.$$

3. We have

$$\frac{\partial x}{\partial u} = \frac{2u}{w}, \quad \frac{\partial x}{\partial w} = -\frac{u^2}{w^2},$$

and

$$\frac{\partial y}{\partial u} = -\frac{w}{u^2}, \quad \frac{\partial y}{\partial w} = \frac{1}{u}.$$

So

$$\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \end{bmatrix} = \det\begin{bmatrix} \frac{2u}{w} & -\frac{u^2}{w^2} \\ -\frac{w}{u^2} & \frac{1}{u} \end{bmatrix} = \frac{1}{w}.$$

So $d(x,y) = \left|\frac{1}{w}\right| d(u,w)$.

Since $xy = u$ and $xy^2 = w$,
the conditions $1 \leq xy \leq 2$ and $3 \leq xy^2 \leq 4$ become $1 \leq u \leq 2$ and $3 \leq w \leq 4$. So

$$\int_D x\, d(x,y) = \int_3^4 \int_1^2 \frac{u^2}{w} \left| \frac{1}{w} \right| du\, dw = \int_3^4 \int_1^2 \frac{u^2}{w^2}\, du\, dw = 0.194444.$$

4(i) $\mathrm{div}(F) = \dfrac{\partial(x-z)}{\partial x} + \dfrac{\partial(yz)}{\partial y} + \dfrac{\partial(xy^2 z)}{\partial z} = 1 + z + xy^2.$

4(ii)

$$\begin{aligned} \mathrm{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & yz & xy^2 z \end{vmatrix} \\ &= (2xyz - y)i - (y^2 z + 1)j + 0k = (2xyz - y)i - (y^2 z + 1)j. \end{aligned}$$

4(iii) $F(r(t)) = \begin{bmatrix} t^2 - 2t - 3 \\ 2t^2 \\ 2t^3(t^2 - 3) \end{bmatrix}$ and $r'(t) = \begin{bmatrix} 2t \\ 1 \\ 2 \end{bmatrix}$ so

$$\int_C F \cdot dr = \int_0^2 2t(t^2 - 2t - 3) + 2t^2 + 4t^3(t^2 - 3)\, dt = -14.666667.$$

5(i) Take the parametrization $r(t) = \begin{bmatrix} t \\ t^2 + 2 \end{bmatrix}$ for $0 \leq t \leq 3$. We have $r'(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$ so $\|r'(t)\| = \sqrt{1 + 4t^2}$. So

$$\int_C x\, d(x,y) = \int_0^3 t\sqrt{1 + 4t^2}\, dt = 18.671851.$$

5(ii) Since $r'(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$, we have $N(t) = \begin{bmatrix} 2t \\ -1 \end{bmatrix}$. Also $F(r(t)) = \begin{bmatrix} 2t^2 + 2 \\ t^3 \end{bmatrix}$. So the flux is

$$\int_0^3 \begin{bmatrix} 2t^2 + 2 \\ t^3 \end{bmatrix} \cdot \begin{bmatrix} 2t \\ -1 \end{bmatrix} dt = \int_0^3 2t(2t^2 + 2) - t^3\, dt = 78.75.$$