Midterm 1 Answer

1(i) $r = \sqrt{2^2 + (-5)^2} = 5.385165$.
Since $x \geq 0$ and $y \leq 0$, we have $\frac{3\pi}{2} \leq \theta \leq 2\pi$.
Calculator gives $\tan^{-1}(-\frac{5}{2}) = -1.190290$.
Correct answer for θ is $-1.190290 + 2\pi = 5.092895$.

1(ii) Let $f(x,y) = \frac{x+1}{y+1}$. We have

$$\begin{aligned} S_{2,3} &= \frac{3-0}{2} \cdot \frac{1-0}{3} \left(f(0,0) + f\left(0, \frac{1}{3}\right) + f\left(0, \frac{2}{3}\right) + f\left(\frac{3}{2}, 0\right) + f\left(\frac{3}{2}, \frac{1}{3}\right) + f\left(\frac{3}{2}, \frac{2}{3}\right) \right) \\ &= \frac{1}{2}\left(1 + \frac{3}{4} + \frac{3}{5} + \frac{5}{2} + \frac{15}{8} + \frac{3}{2}\right) = 4.1125. \end{aligned}$$

1(iii) The curve contains the point $x = y = 0$. Answer: $x^2 + y^2 = \sqrt{x^2 + y^2} + 3x$.

1(iv) $\displaystyle\int_0^2 \int_0^1 (x^2 + y)^2 \, dx \, dy = \int_0^2 \int_0^1 x^4 + 2x^2 y + y^2 \, dx \, dy = \int_0^2 \frac{1}{5} + \frac{2}{3} y + y^2 \, dy = 4.4.$

2(i)

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{[y^2+1,3]} x \, dx \, dy \\ =& \int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2+1}^{3} x \, dx \, dy \\ =& \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{2}(9 - (y^2+1)^2) \, dy = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{2}(8 - y^4 - 2y^2) \, dy = 8\sqrt{2} - \frac{\sqrt{2}^5}{5} - \frac{2 \cdot \sqrt{2}^3}{3} = 8.296720. \end{aligned}$$

2(ii)

$$\begin{aligned} D &= \{(x,y) | \, x \leq 3, \; y^2 \leq x - 1\} \\ &= \{(x,y) | \, x \leq 3, \; x - 1 \geq 0, \; y^2 \leq x - 1\} = \{(x,y) | \, 1 \leq x \leq 3, \; -\sqrt{x-1} \leq y \leq \sqrt{x-1}\}. \end{aligned}$$

$$\begin{aligned} &\int_1^3 \int_{-\sqrt{x-1}}^{\sqrt{x-1}} x \, dy \, dx \\ =& \int_1^3 2x\sqrt{x-1} \, dx \\ =& \left. \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} \right|_{x=1}^{x=3} = 8.296720. \end{aligned}$$

3(i)

$$\begin{aligned} D &= \{(r\cos\theta, r\sin\theta) |\ \sin\theta > 0,\ 0 < r \leq \sqrt{3}\} \\ &= \{(r\cos\theta, r\sin\theta) | \, 0 \leq \theta \leq \pi,\ 0 < r \leq \sqrt{3}\}. \end{aligned}$$

$$\int_0^{\pi} \int_0^{\sqrt{3}} (r\sin\theta) r \, dr \, d\theta = \int_0^{\pi} \int_0^{\sqrt{3}} r^2 \sin\theta \, dr \, d\theta = \int_0^{\pi} \sqrt{3} \sin\theta \, d\theta = 2\sqrt{3} = 3.464101.$$

3(ii)

$$\begin{aligned} D &= \{(x,y) | \, y > 0,\ x^2 \leq 3 - y^2\} = \{(x,y) | \, y > 0,\ 3 - y^2 \geq 0,\ x^2 \leq 3 - y^2\} \\ &= \{(x,y) | \, 0 < y \leq \sqrt{3},\ -\sqrt{3-y^2} \leq x \leq \sqrt{3-y^2}\}. \end{aligned}$$

$$\int_0^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} y \, dx \, dy = \int_0^{\sqrt{3}} 2y\sqrt{3-y^2} \, dy = -\frac{2}{3}(3-y^2)^{\frac{3}{2}} \Big|_{y=0}^{y=\sqrt{3}} = \frac{2}{3} 3^{\frac{3}{2}} = 2\sqrt{3} = 3.464101.$$

4(i)

$$\int_W 1 \, d(x,y,z) = \int_D \int_{x+y}^{3} 1 \, dz \, d(x,y) = \int_D 3 - x - y \, d(x,y),$$

where

$$D = \{(x,y) | \, 0 \leq x \leq 2,\ 0 \leq y \leq 2,\ x + y \leq 3\} = \{(x,y) | \, 0 \leq y \leq 2,\ 0 \leq x \leq \min(2, 3-y)\}.$$

So

$$\int_W 1 \, d(x,y,z) = \int_0^2 \int_0^{\min(2,3-y)} 3 - x - y \, dx \, dy.$$

Since

$$\min(2, 3-y) = \begin{cases} 2, & 2 \leq 3 - y \\ 3 - y, & 3 - y < 2 \end{cases} = \begin{cases} 2, & y \leq 1 \\ 3 - y, & y > 1 \end{cases},$$

we have

$$\begin{aligned} \int_W 1 \, d(x,y,z) &= \int_0^1 \int_0^2 3 - x - y \, dx \, dy + \int_1^2 \int_0^{3-y} 3 - x - y \, dx \, dy \\ &= \int_0^1 4 - 2y \, dy + \int_1^2 \frac{1}{2}(3-y)^2 \, dy = 3 + \frac{7}{6} = 4.166667. \end{aligned}$$

4(ii)

$$\int_W 1 \, d(x,y,z) = \int_D \int_x^y 1 \, dz \, d(x,y) = \int_D y - x \, d(x,y),$$

where $D = \{(x,y) | \, 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ x \leq y\} = \{(x,y) | \, 0 \leq y \leq 2,\ 0 \leq x \leq \min(1,y)\}$.
So

$$\int_W 1 \, d(x,y,z) = \int_0^2 \int_0^{\min(1,y)} y - x \, dx \, dy.$$

Since

$$\min(1,y) = \begin{cases} 1, & y \geq 1 \\ y, & y < 1 \end{cases},$$

we have

$$\begin{aligned} \int_W 1 \, d(x,y,z) &= \int_0^1 \int_0^y y - x \, dx \, dy + \int_1^2 \int_0^1 y - x \, dx \, dy \\ &= \int_0^1 \frac{1}{2} y^2 \, dy + \int_1^2 y - \frac{1}{2} \, dy = \frac{1}{6} + 1 = 1.166667. \end{aligned}$$