Total score: 37 points

- Write your solutions on some papers. Show all your work. Scan as a pdf/jpg file(s). Upload the pdf/jpg file(s) as CCLE Assignment Final before the end time.
- Open book. Calculators are not prohibited. But you cannot get any help from other people.
- You may compute any integral using Fundamental Theorem of Calculus without using the definition involving Riemann sum.
- If your final answer is a number, have it in 5 decimal places.
- You may use calculator to compute an integral like $\int_0^1 \int_0^{2x} \int_x^{x+y} x+y+z\,dz\,dy\,dx$ without showing your work for computing such integral. But you cannot use calculator to compute an integral like $\int_D 1\,d(x,y)$ or $\int_0^1 \int_0^1 \int_{[x,y]} x+y+z\,dz\,dy\,dx$ or an integral involving min/max without showing your work.
- For questions 2, 3(iii), 3(iv), 4, 5, if your set up (that can be put into calculator) is correct and your final answer happens to be wrong, then 0.5 point deducted, regardless of whether or not you use calculator.
- You may use calculator to compute the cross product of two vectors of length 3 without showing your work for computing the cross product.
- **Recommended: After submission, logout and log in CCLE. See if your file is there; download the file you submitted and check if it is the file you intended to submit.**

1. (8 points) For each of the following vector fields, determine whether or not it is conservative. If it is conservative, find the potential function. If it is not conservative, justify your answer.

   (i) $F(x,y,z) = \begin{bmatrix} 2xy + yz \\ x^2 + xz \\ xy \end{bmatrix}$ with domain $\mathbb{R}^3$.

   (ii) $F(x,y) = \begin{bmatrix} y + xy^3 \\ x + x^2y^2 \end{bmatrix}$ with domain $\mathbb{R}^2$.

   (iii) $F(x,y) = \begin{bmatrix} \dfrac{y+x}{x^2+y^2} \\ \dfrac{y-x}{x^2+y^2} \end{bmatrix}$ with domain $D = \{(x,y) | (x,y) \neq (0,0)\}$.

   (iv) $F(x,y) = \begin{bmatrix} \dfrac{x^3}{(x^4+y^4)^2} \\ \dfrac{y^3}{(x^4+y^4)^2} \end{bmatrix}$ with domain $D = \{(x,y) | (x,y) \neq (0,0)\}$.

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2. (8 points) Consider the triangle $\{(x,y)|\, x,y \geq 0,\ 2x+y \leq 1\}$. Let $\mathcal{C}$ be the boundary of this triangle oriented in the anticlockwise direction. Find $\displaystyle\int_{\mathcal{C}} y\,dx + (x^2+y)\,dy$

   (i) using the definition of line integral directly and without using Green's theorem.

   (ii) using Green's theorem.

3. (9 points) Consider the surface $\mathcal{S} = \{(x,y,z)|\, x^2+y^2+2z^2 = 1\}$ and the parametrization

$$G(z,t) = (\sqrt{1-2z^2}\cos t,\ \sqrt{1-2z^2}\sin t,\ z),$$

   for $-\dfrac{1}{\sqrt{2}} \leq z \leq \dfrac{1}{\sqrt{2}}$ and $0 \leq t \leq 2\pi$.

   (i) Find $N(z,t)$.

   (ii) Find the tangent plane of $\mathcal{S}$ at the point $\begin{bmatrix} 0.7 \\ 0.1 \\ 0.5 \end{bmatrix}$.

   Write your answer in the form $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid ax+by+cz = d \right\}$.

   (iii) Calculate $\displaystyle\int_{\mathcal{S}} x^2\, d(x,y,z)$.

   (iv) Take an orientation on $\mathcal{S}$ so that $n(x,y,z) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} < 0$. Let $F(x,y,z) = \begin{bmatrix} x \\ 2y \\ z \end{bmatrix}$.

   Calculate the flux of F across $\mathcal{S}$.

4. (8 points) Consider the oriented surface $\mathcal{S} = \{(x,y,z)|\, z = x+y,\ x^2+y^2 \leq 1\}$ with unspecified orientation. Let $F(x,y,z) = \begin{bmatrix} z \\ 2x \\ 3y \end{bmatrix}$. Verify Stokes' theorem by finding

   (i) $\left|\displaystyle\int_{\partial\mathcal{S}} F \cdot dr\right|$.

   (ii) $\left|\displaystyle\int_{\mathcal{S}} (\mathrm{curl}(F) \cdot n)\, d(x,y,z)\right|$.

5. (4 points) Let $\mathcal{S}$ be the surface enclosing

$$\mathcal{W} = \{(x,y,z)|\, x,y,z \geq 0,\ 2x+3y+z \leq 2,\ x+y+z \geq 1\},$$

   with n pointing to the outside of $\mathcal{W}$. Let $F(x,y,z) = \begin{bmatrix} x+yz \\ y^2x \\ z^2 \end{bmatrix}$. Find $\displaystyle\int_{\mathcal{S}} (F \cdot n)\, d(x,y,z)$.

   Addendum 17 Mar 2021, 14:35 pm: The condition $z \geq 0$ is redundant so

$$\mathcal{W} = \{(x,y,z)|\, x,y \geq 0,\ 2x+3y+z \leq 2,\ x+y+z \geq 1\}.$$

   You may use this $\mathcal{W}$ instead.

End of exam