Exercise 1 (25 points)

(a) [10 pts.] Let $\mathcal{D}$ be the diamond-shaped region in $\mathbb{R}^2$ with vertices at $(0,0)$, (π,π), $(0,2\pi)$, $(-\pi,\pi)$. Find a map which transforms the region $\mathcal{D}_0 = [0,2\pi]\times[0,2\pi]$ in the (u,v)-plane into $\mathcal{D}$. Make a picture of both $\mathcal{D}_0$ and $\mathcal{D}$.

(b) [15 pts.] Use your answer from part (a) to evaluate the following integral:

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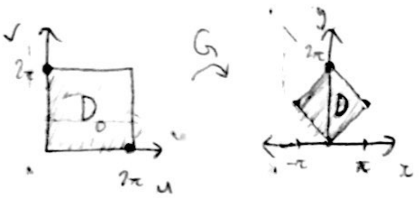
$$\int\int_{\mathcal{D}} (x-y)^2 \sin^2(x+y)\, dA.$$

Hint: the following formula might come in handy: $\sin^2 x = \frac{1-\cos(2x)}{2}$.

a)



Let $x = Au + Cv$

Let $y = Bu + Dv$

$G(0,2\pi) = (2\pi C, 2\pi D) = (-\pi, \pi)$

$G(2\pi,0) = (2\pi A, 2\pi B) = (\pi, \pi)$

$A = \frac{1}{2},\ B = \frac{1}{2},\ C = -\frac{1}{2},\ D = \frac{1}{2}$

$\therefore$ A map transforming D_0 to D is $G(u,v) = \left(\frac{1}{2}u - \frac{1}{2}v, \frac{1}{2}u + \frac{1}{2}v\right)$

b) $\operatorname{Jac}(G) = \det\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$x + y = \frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}u + \frac{1}{2}v = u$

$x - y = \frac{1}{2}u - \frac{1}{2}v - \frac{1}{2}u - \frac{1}{2}v = -v$

Using change of variables,

$I = \int_0^{2\pi}\int_0^{2\pi} (-v)^2 \sin^2 u \cdot \frac{1}{2}\, dv\, du$

$S(u) = \frac{1}{2}\sin^2 u \int_0^{2\pi} v^2\, dv$

$= \frac{1}{2}\sin^2 u \left[\frac{1}{3}v^3\right]_0^{2\pi}$ 2

$= \frac{4\pi^3}{3}\sin^2 u$

$I = \frac{4\pi}{3}\int_0^{2\pi} \frac{1}{2}(1 - \cos 2u)\, du$

$= \frac{2\pi}{3}\int_0^{2\pi} du - \frac{\pi}{3}\int_0^{2\pi} \cos 2u\, 2du$

$= \frac{2\pi}{3}\left[u\right]_0^{2\pi} - \frac{\pi}{3}\left[\sin 2u\right]_0^{2\pi}$

$= \frac{4\pi^2}{3} - 0 = \boxed{\frac{4\pi^3}{3}}$

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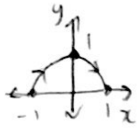
Exercise 2 (24 points)

(a) [12 pts.] Find the total charge on the upper semicircle $x^2 + y^2 = 1$, $y \geq 0$, oriented clockwise, with charge density $\delta(x, y) = xy^3$;
(b) [12 pts.] Find the flux of the vector field $\mathbf{F} = \left(\frac{y^3}{[(x+2)^4+y^4]^{1/2}}, \frac{(x+2)^3}{[(x+2)^4+y^4]^{1/2}}\right)$ across the segment $1 \leq x \leq 3$ oriented left to rigth.

a)



$\vec{r}(t) = \langle -\cos t, \sin t \rangle$, $0 \leq t \leq \pi$

$\vec{r}\,'(t) = \langle \sin t, \cos t \rangle$, $\|\vec{r}\,'(t)\| = 1$

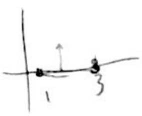
$\delta(\vec{r}(t)) = -\cos t \sin^3 t$

$$\text{Charge} = \int_0^\pi -\cos t \sin^3 t \cdot 1\, dt$$

$$= -\int_0^\pi \sin^3 t \cos t\, dt$$

$$= -\left[\frac{1}{4}\sin^4 t\right]_0^\pi = 0$$

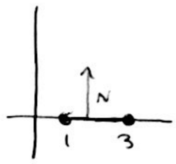
$\boxed{\therefore \text{ the total charge is } 0.}$



b) $\vec{r}(t) = \langle t, 0 \rangle$, $1 \leq t \leq 3$

$\vec{r}\,'(t) = \langle 1, 0 \rangle.$

$$\vec{F}(\vec{r}(t)) = \left\langle 0, \frac{(t+2)^3}{(t+2)^2} \right\rangle = \langle 0, t+2 \rangle$$



Tangential component: $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle 0, t+2 \rangle \cdot \langle 1, 0 \rangle = 0$

So the Normal component of $\vec{F}$ must be equal to $\|\vec{F}\|$ i.e. $\|\langle 0, t+2 \rangle\| = t+2$

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Normal vector is perp. to this segment, so $\text{proj}_{\vec{T}} \vec{F} = 0$ and $\text{proj}_{\vec{N}} \vec{F} = \|\vec{F}\|$

$$I = \int_1^3 t+2\, dt$$

$$= \left[\frac{1}{2}t^2 + 2t\right]_1^3$$

$$= \frac{9}{2} + 6 - \frac{1}{2} - 2$$

$$= 4 + 4 = 8$$

$\boxed{\text{Flux} = 8}$

Exercise 3 (26 points)

(a) [10 pts.] Let $\mathbf{F}(x,y) = \left(\frac{-y+x}{x^2+y^2}, \frac{x+y}{x^2+y^2}\right)$ be a planar vector field. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle $x^2+y^2=4$ oriented counterclockwise.
(b) [8 pts.] Is $\mathbf{F}$ conservative on $\mathcal{D} = \{(x,y) \neq (0,0)\}$? Explain.
(c) [4 pts.] Show that $\mathbf{F}$ satisfies the cross-partial condition.
(d) [4 pts.] Show that $\mathbf{F}$ is conservative on $\mathcal{D} = \{(x,y) | x > 0\}$

a) $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle, \quad 0 \le t \le 2\pi$

$\vec{r}'(t) = \langle -2\sin t, 2\cos t \rangle$

$\vec{F}(\vec{r}(t)) = \left\langle \frac{-2\sin t + 2\cos t}{4}, \frac{2\cos t + 2\sin t}{4} \right\rangle = \left\langle \frac{\cos t - \sin t}{2}, \frac{\cos t + \sin t}{2} \right\rangle$

$I = \int_0^{2\pi} \left\langle \frac{\cos t - \sin t}{2}, \frac{\cos t + \sin t}{2} \right\rangle \cdot \langle -2\sin t, 2\cos t \rangle$

$= \int_0^{2\pi} -\sin t\cos t + \sin^2 t + \cos^2 t + \sin t\cos t$

$= \int_0^{2\pi} 1 \, dt = [t]_0^{2\pi} = \boxed{2\pi}$ ok

consider this curve C, which cannot be pulled tight if there is a hole at $(x,y) \notin (0,0)$

b) No, it is not conservative. There is still a hole in the domain at $(x,y)=0$, so it cannot be simply connected. It follows that $\vec{F}$ cannot be conservative on this domain. Note that the above question is a line integral whose path is on this domain where $(x,y) \neq (0,0)$. If $\vec{F}$ were conservative on this domain, then the answer to part a) could not be 2π. It would be 0.

ok!!

$2x^2+2xy-x^2-y^2$

$\frac{(x+y)(2x) - x^2 - y^2}{(x^2+y^2)^2}$

$(-y+x)(2y) + x^2+y^2$

$-2y^2+2xy+x^2+y^2$

c) $\text{curl}(\vec{F}) = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{-y+x}{x^2+y^2} & \frac{x+y}{x^2+y^2} & 0 \end{bmatrix} = \left\langle 0-0, 0-0, \frac{x^2+2xy-y^2}{(x^2+y^2)^2} - \frac{x^2+2xy-y^2}{(x^2+y^2)^2} \right\rangle = \vec{0}$

$\therefore \text{Curl}(\vec{F}) = \vec{0}$,

$\therefore \vec{F}$ must satisfy the cross-partial condition ok

d) $\vec{F}$ is defined on all of $\mathbb{R}^2$ where $D = \{(x,y) | x > 0\}$ which is a simply connected domain

Since $\vec{F}$ satisfies the cross-partial condition,

$\therefore \vec{F}$ must be conservative on this domain. ok

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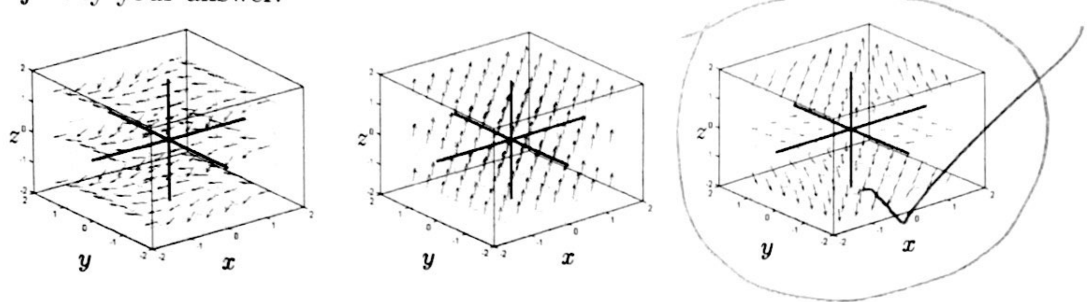
Exercise 4 (25 points)

(a) [5 pts.] Given the three-dimensional vector field $\mathbf{F}(x,y,z) = (\frac{y}{1+x^2}, \tan^{-1} x, 2z)$ which of the following is a plot of $\mathbf{F}$? Circle the right one, you do not need to justify your answer.

$y(1+x^2)^{-1}$

$-y\,(1+x^2)^{-2} \cdot 2x$



(b) [5 pts.] Compute div($\mathbf{F}$) and curl($\mathbf{F}$);
(c) [15 pts.] Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ over the unit circle in the (x,y)-plane clockwise oriented.

b) $\operatorname{div}(\vec{F}) = -\dfrac{2xy}{(1+x^2)^2} + 0 + 2 = 2 - \dfrac{2xy}{(1+x^2)^2}$

$$\operatorname{curl}(\vec{F}) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{y}{1+x^2} & \tan^{-1}x & 2z \end{bmatrix} = \left\langle 0-0, -(0-0), \frac{1}{1+x^2} - \frac{1}{1+x^2} \right\rangle = \vec{0}$$

c) $\vec{F}$ satisfies the cross-partials condition and all of its components are defined for all of $\mathbb{R}^3$, which is a simply connected domain. So $\vec{F}$ must be conservative.
Since C is a closed curve,

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = 0.}$$