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MATH 32B Midterm I, Fall 2018

Circle Your TA's Name and Section Number: Adam Lott 2A 2B, Bumsu Kim 2C 2D, Derek Levinson 2E 2F

Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

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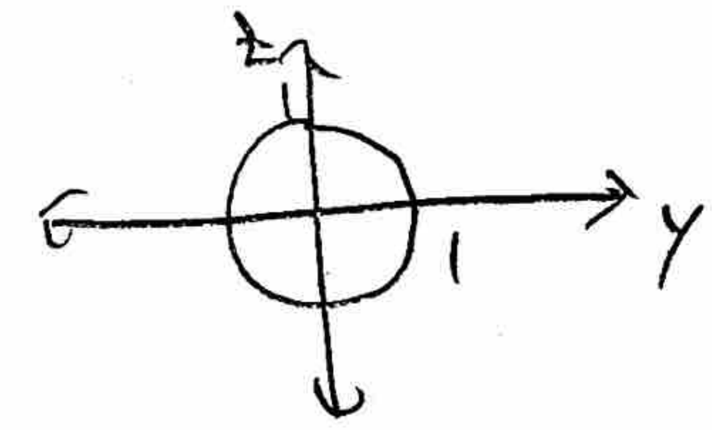
Problem 1. (4)

Find the triple integral $\int\int\int_W (y^2+z^2)\,dxdydz$. Here W is the finite solid bounded by the surface $x = 1 - y^2 - z^2$ and the plane $x = 0$.

on yz plane: $0 = 1 - y^2 - z^2$

$y^2 + z^2 = 1$



$x = 1 - y^2 - z^2 \to 1 - r^2$

$0 \le x \le 1 - r^2$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$y^2 + z^2 \to r^2$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{x=0}^{1-r^2} r^2 \, r\,dx\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r^3\,dx\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3 x\Big|_0^{1-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3(1-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3 - r^5\,dr\,d\theta = \int_0^{2\pi} \frac{r^4}{4} - \frac{r^6}{6}\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} - \frac{1}{6}\,d\theta = \int_0^{2\pi} \frac{3}{12} - \frac{2}{12}\,d\theta = \int_0^{2\pi} \frac{1}{12}\,d\theta = 2\pi\left(\frac{1}{12}\right) = \frac{2\pi}{12}$$

$$= \boxed{\frac{\pi}{6}}$$

Problem 2. (4)

Compute the center of mass of D, where D is the finite region in $\mathbf{R}^2$ with $y \geq 0$ and bounded by $y = x$, $y = -x$, $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Here the density function $\rho(x, y) = y^2$.

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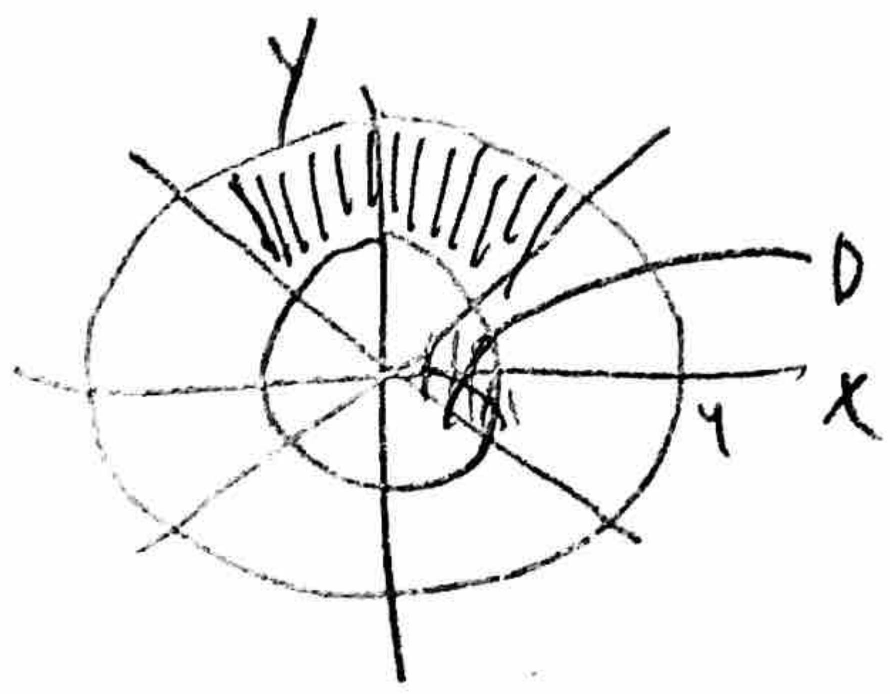

$0 \leq r \leq 1$ $\qquad$ $\rho(x,y) = y^2 = r^2 \sin^2\theta$

$-\frac{\pi}{4} \leq \theta \leq \pi/4$

$$\text{Mass} = \int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{1} r^2 \sin^2\theta \, r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \int_0^1 r^3 \sin^2\theta \, dr \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{r^4}{4} \sin^2\theta \Big|_0^1 d\theta = \frac{1}{4} \int_{-\pi/4}^{\pi/4} \sin^2\theta \, d\theta = \frac{1}{4} \int_{-\pi/4}^{\pi/4} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \frac{1}{8} \int_{-\pi/4}^{\pi/4} 1 - \cos 2\theta \, d\theta = \frac{1}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_{-\pi/4}^{\pi/4} = \frac{1}{8}\left[\left(\frac{\pi}{4} - \frac{\sin(\pi/2)}{2}\right) - \left(-\frac{\pi}{4} - \frac{\sin(-\pi}{2}\right.\right.$$

$$= \frac{1}{8}\left[\frac{\pi}{4} - \frac{1}{2} - \left(-\frac{\pi}{4} + \frac{1}{2}\right)\right] = \frac{1}{8}\left[\frac{\pi}{2} - 1\right] = \frac{1}{8}\left(\frac{\pi}{2} - 1\right) \quad \text{Mass} = \frac{1}{8}\left(\frac{\pi}{2} - \right.$$

$$\int_{-\pi/4}^{\pi/4} \int_0^1 r\cos\theta \, r^2 \sin^2\theta \, r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \int_0^1 r^4 \cos\theta \sin^2\theta \, dr \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{r^5}{5} \cos\theta \sin^2\theta \Big|_0^1 d\theta = \frac{1}{5} \int_{-\pi/4}^{\pi/4} \cos\theta \sin^2\theta \, d\theta$$

let $u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= \frac{1}{5} \int_{-\sqrt{2}/2}^{\sqrt{2}/2} u^2 \, du = \frac{1}{5}\left[\frac{u^3}{3}\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = \frac{1}{5}\left[\frac{2\sqrt{2}}{8} \cdot \frac{1}{3} - \left(-\frac{2\sqrt{2}}{8} \cdot \frac{1}{3}\right)\right]$$

$$= \frac{1}{5}\left[\frac{\sqrt{2}}{12} + \frac{\sqrt{2}}{12}\right] = \frac{1}{5}\left[\frac{\sqrt{2}}{6}\right] = \frac{\sqrt{2}}{30}$$

$$X_{cm} = \frac{\frac{\sqrt{2}}{30}}{\frac{1}{8}\left(\frac{\pi}{2} - 1\right)} = \frac{8\sqrt{2}}{30(\pi/2 - 1)}$$

$Y_{cm} = 0$ by symmetry

$$\text{CM: } \left(\frac{8\sqrt{2}}{30(\pi/2 - 1)}, 0\right)$$

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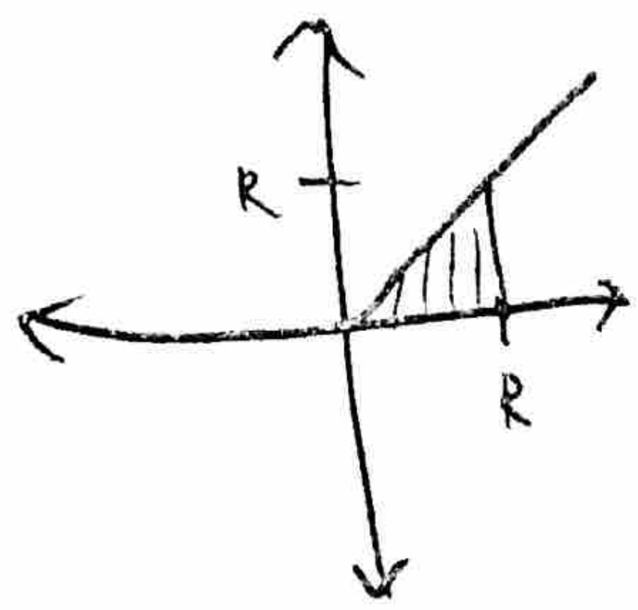


Problem 3. (4)

Find the iterated integral $\int_0^R \int_{x=y}^R ycos(x^3)dxdy$. Here R is a positive constant. $R > 0$

Hint: Convert the iterated integral into a double integral and evaluate the double integral.

$0 \le y \le x$

$0 \le x \le R$

$$\int_{x=0}^{R} \int_{y=0}^{x} y\cos(x^3)\,dy\,dx = \int_0^R \frac{y^2}{2}\cos(x^3)\Big|_0^x \,dx$$

$$= \int_0^R \frac{x^2}{2}\cos(x^3)\,dx$$

let $u = x^3$

$du = 3x^2\,dx$

$$= \frac{1}{6}\int_0^{R^3} \cos(u)\,du = \frac{1}{6}\Big[\sin(u)\Big]_0^{R^3}$$

$$= \frac{1}{6}\Big[\sin(R^3) - \sin(0)\Big] = \frac{1}{6}\Big[\sin(R^3) - 0\Big]$$

$$= \boxed{\frac{\sin(R^3)}{6}}$$

Problem 4. (4)

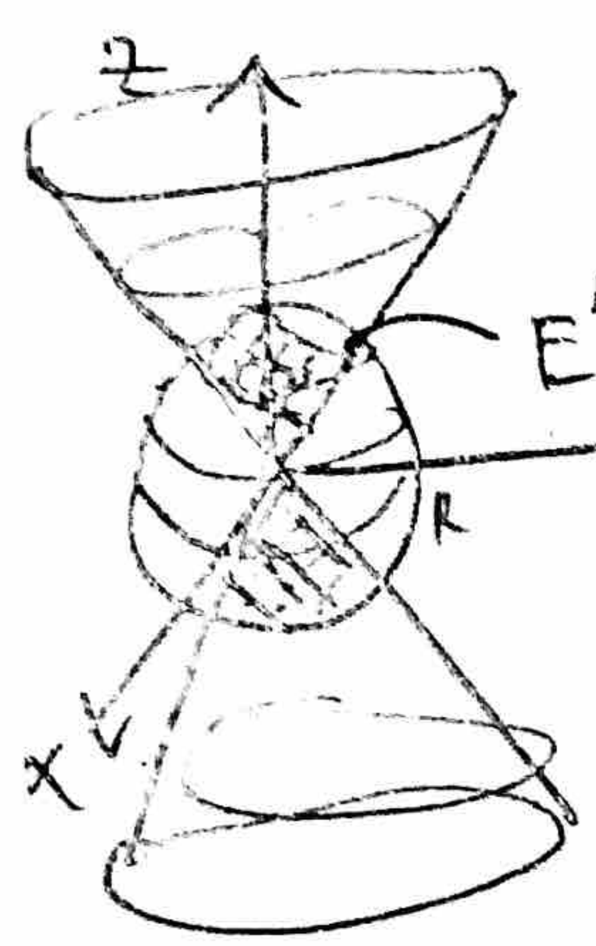
Find

by symmetry

$$\iiint_E x^2+y^2\,dxdydz, \quad = 2\iiint_{E'} x^2+y^2\,dx\,dy\,dz$$

where E is the finite solid bounded by the sphere $x^2+y^2+z^2=R^2$ and the cone $x^2+y^2-z^2=0$. Here R is a positive constant.
Note : E is inside both the sphere and the cone.

$R>0$



$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

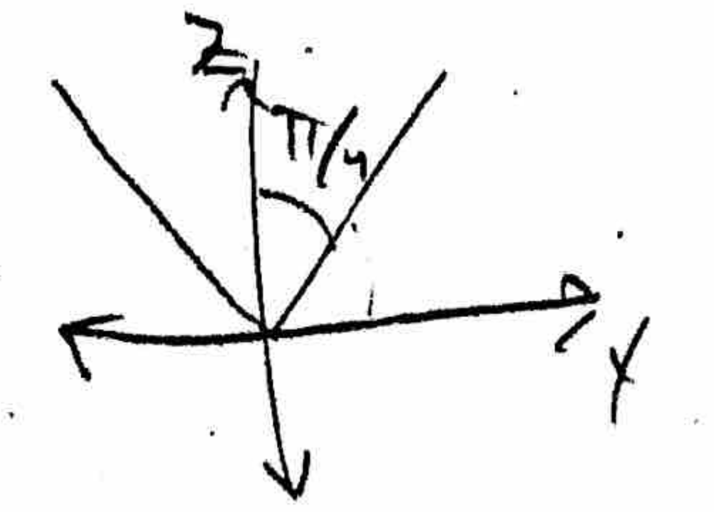
$z = \rho\cos\phi$

$\rho^2 = R^2$

$0 \le \rho \le R$

$0 \le \phi \le \pi/4$

$0 \le \theta \le 2\pi$



$x^2+y^2 = \rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta)$

$= \rho^2 \sin^2\phi$

$$2\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{R} \rho^2\sin^2\phi\ \rho^2\sin\phi\ d\rho\, d\phi\, d\theta \quad \checkmark$$

$$= 2\int_0^{2\pi}\int_0^{\pi/4}\int_0^R \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta = 2\int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^5}{5}\sin^3\phi \Big|_0^R\, d\phi\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^{\pi/4}\frac{R^5}{5}\sin^3\phi\, d\phi\, d\theta = \frac{2R^5}{5}\int_0^{2\pi}\int_0^{\pi/4}(1-\cos^2\phi)\sin\phi\, d\phi\, d\theta$$

let $u = \cos\phi$
$du = -\sin\phi\, d\phi$

$$= \frac{2R^5}{5}\int_0^{2\pi}\int_1^{\sqrt{2}/2} -1(1-u^2)\, du\, d\theta = \frac{2R^5}{5}\int_0^{2\pi}\int_{\sqrt{2}/2}^1 1-u^2\, du\, d\theta$$

$$= \frac{2R^5}{5}\int_0^{2\pi} u - \frac{u^3}{3}\Big|_{\sqrt{2}/2}^1 d\theta = \frac{2R^5}{4\,5}\int_0^{2\pi}\left(1-\frac{1}{3}\right) - \left(\frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{8}\cdot\frac{1}{3}\right) d\theta$$

$$= \frac{2R^5}{5}\int_0^{2\pi}\frac{2}{3} - \left(\frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{24}\right)d\theta = \frac{4\pi R^5}{5}\left(\frac{2}{3} - \left(\frac{12\sqrt{2}}{24} - \frac{2\sqrt{2}}{24}\right)\right)$$

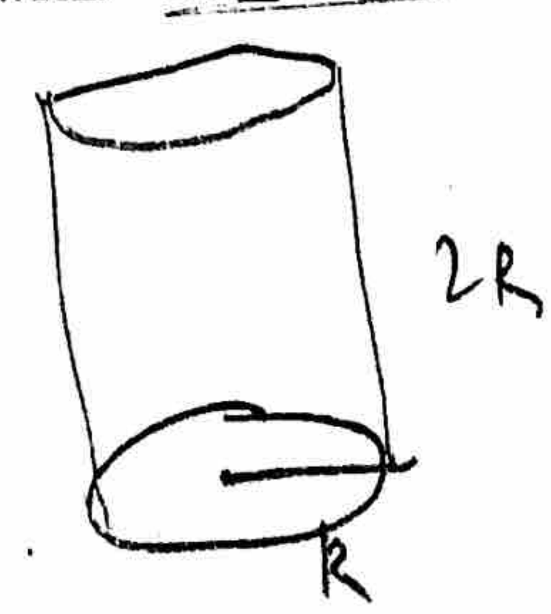
$$= \frac{4\pi R^5}{5}\left(\frac{2}{3} - \frac{10\sqrt{2}}{24}\right) = \frac{\cancel{4}\pi R^5}{5}\left(\frac{\cancel{16}^{\,8} - \cancel{10}^{\,5}\sqrt{2}}{\cancel{24}\,\cancel{6}\,3}\right) = \boxed{\frac{\pi R^5(8-5\sqrt{2})}{15}}$$

Problem 5. (4)

Estimate the following integral

$$\iiint_E e^{(x^2+y^2)\cdot \sin z}\, dV,$$

where E is the solid inside the cylinder $x^2+y^2=R^2$ with $-R \le z \le R$. Here R is a positive constant.



$-1 \le \sin z \le 1$

$x^2+y^2 \le R^2$

$-R^2 \le (x^2+y^2)\sin z \le R^2$

$e^{-R^2} \le e^{(x^2+y^2)\sin z} \le e^{R^2}$

Volume $= \pi r^2 h$

Volume $= \pi (R)^2 (2R)$

Volume $= 2\pi R^3$

Can be estimated by:

$$2\pi R^3 e^{-R^2} \le \iiint_E e^{(x^2+y^2)\sin z}\, dV \le 2\pi R^3 e^{R^2}$$

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