11

1	0.5
2	3
3	4
4	3
5	0.5

MATH 32B Midterm I, Fall 2015

Name: [redacted]

Circle Your TA's Name and Section Number: **Yuming Zhang 2A 2B, Matthew Stoffregen 2C 2D, Qianchang Wang 2E 2F**

Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Find the triple integral

$$\iiint_E x^2\, dV,$$

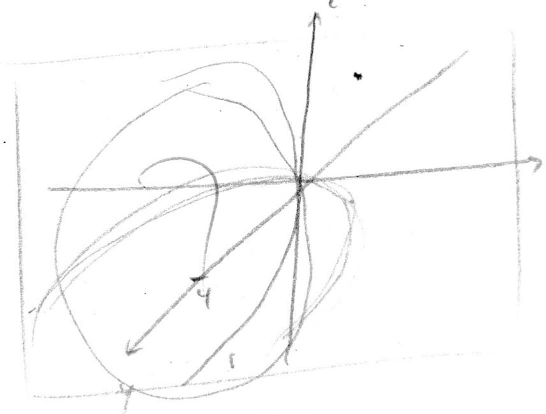
where E is the region bounded by the parabloid $y = 4(x^2 + z^2)$ and the plane $y = 4$.

$x^2+z^2 = \frac{y}{4}$

$0 \le y \le 4$

$-\frac{\sqrt{y}}{2} \le x \le \frac{\sqrt{y}}{2}$

$-\frac{\sqrt{y}}{2} \le z \le \frac{\sqrt{y}}{2}$



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

~~$0 \le y \le 4$~~

$= 4\int_0^4 \int_0^{\frac{\sqrt{y}}{2}} \int_0^{\frac{\sqrt{y}}{2}} x^2\, dz\,dx\,dy$

$4\int_0^4 \int_0^{\frac{\sqrt{y}}{2}} x^2 z \Big|_0^{\frac{\sqrt{y}}{2}} dx\,dy$

$= 4\int_0^4 \int_0^{\frac{\sqrt{y}}{2}} \frac{x^2\sqrt{y}}{2}\, dx\,dy$

$= 2\int_0^4 \frac{1}{3}x^3\sqrt{y}\Big|_0^{\frac{\sqrt{y}}{2}} dy$

$= \frac{2}{3}\int_0^4 x^3\sqrt{y}\Big|_0^{\sqrt{y}/2} dy = \frac{2}{3}\int_0^4 \left(\frac{\sqrt{y}}{2}\right)^3\sqrt{y}\,dy = \frac{2}{3}\int_0^4 \frac{y\sqrt{y}}{8}\sqrt{y}\,dy$

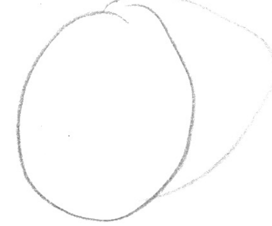
$= \frac{2}{3}\int_0^4 \frac{y^2}{8}\,dy = \frac{1}{12}\int_0^4 y^2\,dy = \frac{1}{36}(y^3)\Big|_0^4 dy$

$= \frac{1}{36}(64-0) = \frac{64}{36} = \frac{32}{18} = \boxed{\frac{16}{9}}$

$y = 4(0+z^2)$

$y = 4x^2$

$x = \pm\frac{\sqrt{y}}{2}$



$y = 4$

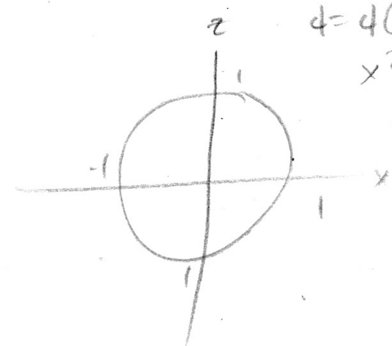
$y = 4(x^2+z^2)$

$4 = 4(x^2+z^2)$

$x^2+z^2 = 1$

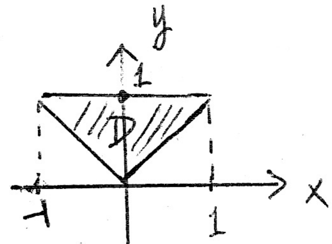
circle

$r = 1$



Problem 2. (4)

Compute the center of of mass of D given by the following picture. Here the density fuction $\rho(x, y) = y^2$.



$0 \le x \le 1$

$x \le y \le 1$

$X_{CM} = \dfrac{\iint x\rho \, dA}{M(D)}$

$M(D) = \iint y^2 \, dA$

$M(D) = \underbrace{2}_{\text{because symmetrical}} \int_0^1 \int_x^1 y^2 \, \cancel{dA} \, dy dx$

$= 2\int_0^1 \frac{1}{3} y^3 \Big|_x^1 \, \cancel{dA} \, dx$

$= \frac{2}{3} \int_0^1 1 - x^3 \, dx$

$= \frac{2}{3}(x - \frac{1}{4}x^4)\Big|_0^1 \, dx$

$= \frac{2}{3}((1 - \frac{1}{4}) - 0) = \frac{2}{3}(1 - \frac{1}{4}) = \frac{2}{3}(\frac{3}{4}) = \frac{1}{2}$ ✓

divide by $M(D) = \frac{1}{2}$ (first quadrant)

$X_{CM_1} = 2\int_0^1 \int_x^1 xy^2 \, dy dx = 2\int_0^1 \frac{1}{3}xy^3 \Big|_x^1 dx = 2\int_0^1 \frac{1}{3}x - \frac{1}{3}x^4 \, dx$

$= \frac{2}{3}\int_0^1 x - x^4 \, dx = \frac{2}{3}(\frac{1}{2}x^2 - \frac{1}{5}x^5 \Big|_0^1)$ $= \frac{2}{3}((\frac{1}{2} - \frac{1}{5}) - 0) = \frac{2}{3}(\frac{5}{10} - \frac{2}{10}) = \frac{2}{3}(\frac{3}{10}) = \frac{1}{5}$

$X_{CM_2} = 2\int_{-1}^0 \int_{-x}^1 xy^2 \, dy dx = 2\int_{-1}^0 \frac{1}{3}xy^3 \Big|_{-x}^1 dx = 2\int_{-1}^0 \frac{1}{3}x + \frac{1}{3}x^4 \, dx$

$= \frac{2}{3}(\frac{1}{2}x^2 + \frac{1}{5}x^5 \Big|_{-1}^0) = \frac{2}{3}(0 - (\frac{1}{2} - \frac{1}{5})) = \frac{-2}{3}(\frac{5}{10} - \frac{2}{10}) = \frac{-2}{3}(\frac{3}{10}) = -\frac{1}{5}$

$\frac{1}{5} + (-\frac{1}{5}) = 0 = X_{CM}$ & symmetry

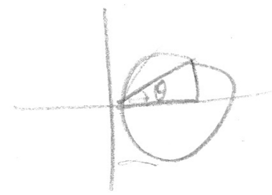
$Y_{CM_1} = 2\int_0^1 \int_x^1 y^3 \, dy dx = 2\int_0^1 \frac{1}{4}y^4 \Big|_x^1 dx = \frac{1}{2}\int_0^1 y^4 \Big|_x^1 dx$

$= \frac{1}{2}\int_0^1 1 - x^4 \, dx = \frac{1}{2}(x - \frac{1}{5}x^5)\Big|_0^1 = \frac{1}{2}((1 - \frac{1}{5}) - 0) = \frac{1}{2}(\frac{4}{5}) = \frac{4}{10} = \frac{2}{5}$

$Y_{CM_2} = 2\int_{-1}^0 \int_{-x}^1 y^3 \, dy dx = 2\int_{-1}^0 \frac{1}{4}y^4 \Big|_{-x}^1 dx = \frac{1}{2}\int_{-1}^0 y^4 \Big|_{-x}^1 dx = \frac{1}{2}\int_{-1}^0 1 - x^4 \, dx$ same

$= \frac{1}{2}(x - \frac{1}{5}x^5)\Big|_{-1}^0 = \frac{1}{2}(0 - (-1 + \frac{1}{5})) = -\frac{1}{2}(-\frac{4}{5}) = \frac{4}{10} = \frac{2}{5}$

Center of Mass $= \boxed{(0, \frac{2}{5})}$



$P = r\cos\theta$

Problem 3. (4)

Find the iterated integral $\int_0^R \int_0^{\sqrt{R^2-x^2}} e^{(x^2+y^2)}\,dy\,dx$. Here R is a positive constant.

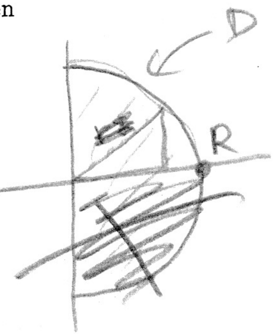
Hint: Convert the iterated integral into a proper double integral and then evaluate the double integral.

$x^2+y^2=R^2$ $x^2+y^2=r^2$ $\sqrt{x^2+y^2}$

$y=\sqrt{R^2-x^2}$

$y=\sqrt{R^2-x^2}$

$y=\sqrt{R^2-r^2\cos^2\theta}$



$=\int_0^{\pi/2}\int_0^{R\cos\theta} e^{r^2}\cdot r\,dr\,d\theta$

$=\int_0^{\pi/2}\int_0^{R\cos\theta} re^{r^2}\,dr\,d\theta$

$u=r^2$

$du=2r\,dr$

$dr=\frac{du}{2r}$

$0\le\theta\le\frac{\pi}{2}$

$0\le r\le R\cos\theta$

$0\le r\le R$

$=\int_0^{\pi/2}\int_0^{R\cos\theta} r\cdot\frac{1}{2r}e^u\,du\,d\theta = \frac{1}{2}\int_0^{\pi/2} e^u\Big|_0^{R\cos\theta}\,d\theta$

$=\frac{1}{2}\int_0^{\pi/2} e^{r^2}\Big|_0^{R\cos\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(e^{R^2\cos^2\theta}-1\right)d\theta$

$=\frac{1}{2}\int_0^{\pi/2} e^{R^2-R^2\sin^2\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{e^{R^2}}{e^{R^2\sin^2\theta}}\,d\theta$

$=\frac{1}{2}\int_0^{\pi/2}$

$R^2\cos^2\theta = R^2-R^2\sin^2\theta$

$=R^2-R^2\sin^2\theta$

$\int_0^{\frac{\pi}{2}}\int_0^R re^{r^2}\,dr\,d\theta = \int_0^{\pi/2}\int_0^R r\cdot\frac{1}{2r}e^u\,du\,d\theta = \frac{1}{2}\int_0^{\pi/2} e^{r^2}\Big|_0^R\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(e^{R^2}-1\right)d\theta$

$=\frac{1}{2}\cdot\frac{\pi}{2}\left(e^{R^2}-1\right)$

$=\boxed{\frac{\pi}{4}e^{R^2}-\frac{\pi}{4}}$

$x^2+y^2+z^2=\rho^2$

$\rho=R$

ρ 3

Problem 4. (4)

Find the triple integral

$$\iiint_W x^2\, dV,$$

where W is the region in $\mathbf{R}^3$ bounded by the sphere $x^2+y^2+z^2=R^2$ of the radius R and the cone $x^2+y^2=z^2$ with $z \geq 0$.

$x^2+y^2=R^2-z^2$ gives definite value

sphere $\Rightarrow x^2+y^2+z^2=R^2$

$\iiint_W x^2\, dV$

$x^2+y^2+z^2=R^2$

$x^2+y^2=z^2$

$x^2+y^2-z^2=0$

$x^2+y^2=z^2$

$= \iiint_W (\rho\sin\varphi\cos\theta)^2\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^R \rho^2\sin^2\varphi\cos^2\theta\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^R \rho^4\sin^3\varphi\cos^2\theta\, d\rho\, d\varphi\, d\theta$

$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/4} \rho^5\sin^3\varphi\cos^2\theta \Big|_0^R d\varphi\, d\theta$

$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/4} R^5\sin^3\varphi\cos^2\theta\, d\varphi\, d\theta$

$= \left(\int_0^{2\pi}\cos^2\theta\, d\theta\right)\left(\int_0^{\pi/4}\sin^3\varphi\, d\varphi\right)\left(\int_0^R \rho^4\, d\rho\right)$

how?

$= (\quad)\left(\frac{2}{3}-\frac{5\sqrt{2}}{12}\right)\left(\frac{1}{5}\rho^5\Big|_0^R\right)$

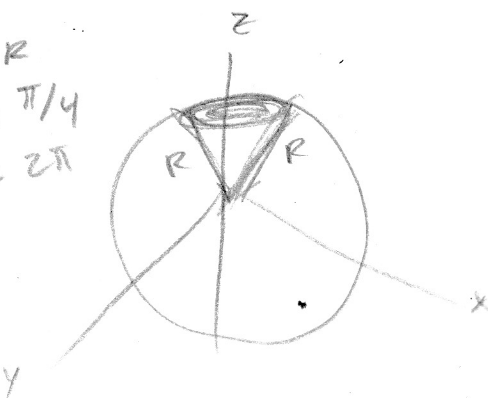
$= (\quad)\left(\frac{2}{3}-\frac{5\sqrt{2}}{12}\right)\left(\frac{R^5}{5}\right)$

$R=$

$0 \leq \rho \leq R$

$0 \leq \varphi \leq \pi/4$

$0 \leq \theta \leq 2\pi$



$\sqrt{x^2+y^2}=z$

$x^2+y^2+z^2=R^2$

$\sqrt{\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta+\rho^2\cos^2\varphi} = R^2$

$\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta=\rho^2\cos^2\varphi$

$\rho^2\sin^2\varphi(1)=\rho^2\cos^2\varphi$

$\sin^2\varphi=\cos^2\varphi$

$\tan\varphi=1$

$\varphi=\pi/4$

$\sqrt{\rho^2\sin^2\varphi+\rho^2\cos^2\varphi} = R^2$

$\rho=R$

0.5 **Problem 5.** (4)

Estimate the following integral

?

$$\int\int\int_E e^{sinxcosysinz}\,dV,$$

where E is the region inside the cylinder $x^2+z^2=1$ between the planes $y=-1$ and $y=2$.

~~$r\cos\theta\, r$~~

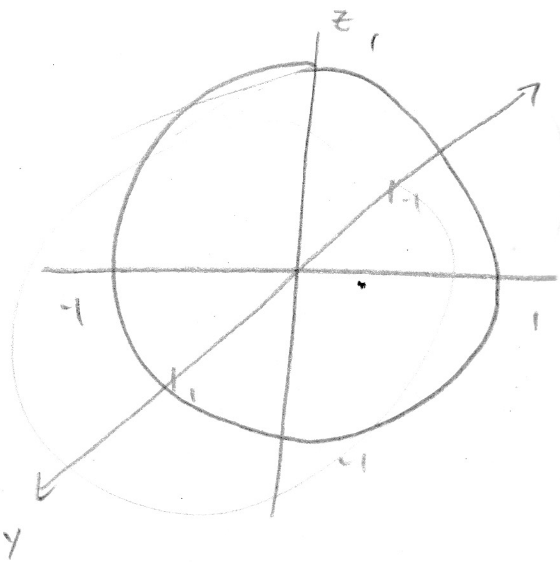
$x^2+z^2=1$

$y=-1$

$y=2$

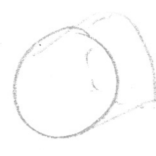
cylindrical coordinates.

$$\int_0^{2\pi}\int_0^1\int_{-1}^2$$



$$\int_0^{2\pi}\int_{-1}^2\int$$

~~dzd~~



$-1 \le y \le 1$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$-1 \le \cancel{z}\, y \le 2$

something