Test 02

MATH 32b @ $\mathscr{U}CLA$ (Winter 2022): Test 02

Assigned: February 24, 2022.

Instructions/Oath

1. SHOW ALL WORK.
   A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit.
2. Duration: 24 hours.
3. The following is my own work, without the aid of any other person.
   Signature: ____

Question 1 ***Change of Variables***

Let $\mathscr{D} = \{(x, y) \in \mathbb{R}^2; |y| + |y - x| \leq 1\}$ and $\mathscr{I} = \displaystyle\iint_{\mathscr{D}} \frac{(2y - x)^2}{x^2 + 3} dA$.

(i) Sketch the graph of the domain $\mathscr{D}$ and then use it to give a change of variables u and v.

(ii) Evaluate the double integral $\mathscr{I}$.

Question 2 ***A conservative Vector Field***

The vector field $\mathbf{F}(x,y) = \left\langle \frac{-x}{(x^2+y^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2)^{\frac{3}{2}}} \right\rangle$ is defined on the region $\mathscr{D} = \{(x,y) \neq (0,0)\}$.

(i) Is $\mathscr{D}$ a simply connected region?

(ii) Show that $\mathbf{F}$ satisfies the cross-partials condition. Does this guarantee that $\mathbf{F}$ is conservative?

(iii) Show that $\mathbf{F}$ is conservative on $\mathscr{D}$ by finding a potential function.

Question 3 *Different forms of Green's Theorem*

Consider the region $\mathscr{R}$ defined by $x^2 - 3 \le y \le 5 - x^2$ and the vector field $\mathbf{F} = \left\langle \overset{P}{xy}, \overset{Q}{xy + 2x} \right\rangle$. Using Green's Theorem

(i) find the counterclockwise circulation of $\mathbf{F}$ around the boundary of $\mathscr{R}$,

(ii) find the outward flux of $\mathbf{F}$ across the boundary of $\mathscr{R}$.

Question 4 ***The Gravitational Potential***

The potential function for the gravitational force field due to a mass M at the origin acting on a mass m is $\phi = \dfrac{GMm}{|\mathbf{r}|}$, where $\mathbf{r} =< x, y, z >$ is the position vector of the mass m and G is the gravitational constant.

(i) Compute the gravitational force $\mathbf{F} = -\nabla\phi$

(ii) Show that the field is irrotational, that is $\nabla \times \mathbf{F} = \mathbf{0}$.