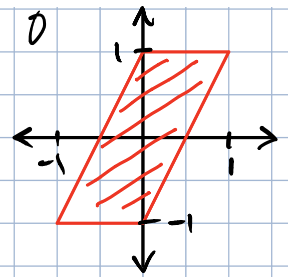
Exercise 1 CHANGE OF VARIABLES.

Let $\mathcal{D} = \{(x,y) \in \mathbb{R}^2; |x| + |y-x| \le 1\}$ and $\mathcal{I} = \iint_{\mathcal{D}} \frac{(y-2x)^2}{y^2+1} dxdy$.

(i) Sketch the graph of the domain $\mathcal{D}$ and then use it to give a change of variables u and v.

(ii) Evaluate the double integral $\mathcal{I}$.

(i) $\mathcal{D}$



$$\left.\begin{matrix} u = y - 2x \\ v = y \end{matrix}\right\} \rightarrow \begin{matrix} y = v \\ x = \dfrac{v-u}{2} \end{matrix}$$

$$2x - 1 \le y \le 2x + 1$$

$$-1 \le y - 2x \le 1 \rightarrow -1 \le u \le 1$$

$$-1 \le y \le 1 \longrightarrow -1 \le v \le 1$$

$$G(u,v) = \left(\frac{v-u}{2}, v \right)$$

$$|Jac(G)| = \begin{vmatrix} \frac{\partial x}{\partial u} = \frac{-1}{2} & \frac{\partial x}{\partial v} = \frac{1}{2} \\ \frac{\partial y}{\partial u} = 0 & \frac{\partial y}{\partial v} = 1 \end{vmatrix} = \left| -\frac{1}{2} \right| = \frac{1}{2}$$

$$\boxed{\iint_D \frac{(y-2x)^2}{y^2+1} = \int_{-1}^{1} \int_{-1}^{1} \frac{u^2}{v^2+1} \cdot \frac{1}{2} \, du \, dv}$$

(ii)

$$\frac{1}{2} \underbrace{\int_{-1}^{1} u^2 \, du}_{\frac{1}{3} u^3 \Big|_{-1}^{1}} \underbrace{\int_{-1}^{1} \frac{1}{v^2+1} dv}$$

$$= \frac{2}{3}$$

$v = \tan\theta$ we know, $\tan^2\theta + 1 = \sec^2\theta$

$dv = \sec^2\theta \, d\theta$

$$\int \frac{1}{\tan^2\theta + 1} \sec^2 d\theta = \int \frac{1}{\cancel{\sec^2\theta}} \cancel{\sec^2\theta} \, d\theta = \theta$$

$\theta = \tan^{-1} v$

$$\tan^{-1}(v) \Big|_{-1}^{1} = \frac{\pi}{2}$$

$$= \frac{-1}{2} \cdot \frac{\cancel{2}}{3} \cdot \frac{\pi}{\cancel{2}} = \boxed{\frac{\pi}{6}}$$

Exercise 2 CONSERVATIVE VECTOR FIELD.

The vector field $\mathbf{F}(x,y) = \left\langle \frac{-x}{(x^2+y^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2)^{\frac{3}{2}}} \right\rangle$ is defined on the region $\mathcal{D} = \{(x,y) \neq (0,0)\}$.

(i) Is $\mathcal{D}$ a simply connected region?

(ii) Show that $\mathbf{F}$ satisfies the cross-partials condition. Does this guarantee that $\mathbf{F}$ is conservative?

(iii) Show that $\mathbf{F}$ is conservative on $\mathcal{D}$ by finding a potential function.

(i) No, as it is not defined at $(x, y) = (0, 0)$.

(ii)

$$F(x,y) = \langle \underbrace{\frac{-x}{(x^2+y^2)^{3/2}}}_{F_1}, \underbrace{\frac{-y}{(x^2+y^2)^{3/2}}}_{F_2} \rangle$$

$$\frac{\partial F_1}{\partial y} = -x \cdot \cancel{2}y \cdot \frac{3}{\cancel{2}} \sqrt{x^2+y^2} = -3xy\sqrt{x^2+y^2}$$

$$= \checkmark$$

$$\frac{\partial F_2}{\partial x} = -y \cdot \cancel{2}x \cdot \frac{3}{\cancel{2}} \sqrt{x^2+y^2} = -3xy\sqrt{x^2+y^2}$$

Satisfies cross-partials. No guarantee, as D is not simply connected.

(iii)

$$f(x,y) = \int \frac{-x}{(x^2+y^2)^{3/2}}\,dx = \frac{1}{\sqrt{x^2+y^2}} + g(y)$$

$$f(x,y) = \int \frac{-y}{(x^2+y^2)^{3/2}}\,dy = \frac{1}{\sqrt{x^2+y^2}} + C$$

$$g(y) = C$$

Potential function: $\boxed{f(x,y) = \frac{1}{\sqrt{x^2+y^2}} + C}$ $\forall\ C \in \mathbb{R}$

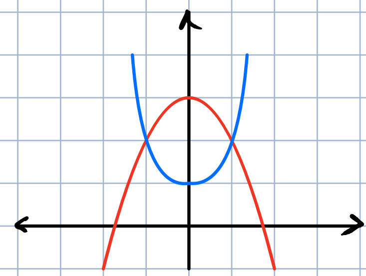
Exercise 3 GREEN's THEOREM.

Consider the region $\mathcal{D}$ bounded above by the curve $y = 3 - x^2$ and below by the curve $y = x^4 + 1$ and the vector field $\mathbf{F} = \left\langle xy, x^2 + x \right\rangle$. Using Green's Theorem

(i) Find the counterclockwise circulation of $\mathbf{F}$ around the boundary of $\mathcal{D}$.

(ii) Find the outward flux of $\mathbf{F}$ accross the boundary of $\mathcal{D}$.

(i)
$$\oint_C F \cdot d\vec{r} = \oint_C (xy)\,dx + (x^2+x)\,dy$$



$$\frac{\partial}{\partial y} F_1 = x$$

$$\frac{\partial}{\partial x} F_2 = 2x - 1$$

$$\iint_D 2x - 1 \, dA = \int_{-1}^{1}\int_{x^4+1}^{3-x^2} 2x - 1 \, dy\, dx = \int_{-1}^{1} (2x-1)\left[y \right|_{x^4+1}^{3-x^2} dx$$

$$= \int_{-1}^{1} (2x-1)(\cancel{3}^{\,2} - x^2 - x^4 \cancel{-1})\, dx$$

$$= \int_{-1}^{1} 4x - 2x^3 - 2x^5 - 2 + x^2 + x^4 \, dx$$

$$= \left[2x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - 2x + \frac{1}{3}x^3 + \frac{1}{5}x^5 \right|_{-1}^{1}$$

only odd powers matter:

$$= \left[-2 + \frac{1}{3} + \frac{1}{5} \right] - \left[2 - \frac{1}{3} - \frac{1}{5} \right]$$

$$= \boxed{\frac{-44}{15}}$$

(ii)
$$\oint_{\partial D} F \cdot n \, ds = \iint_D \operatorname{div}(F)\, dA$$

(contd. on next page)

(ii) contd.

$$\frac{\partial}{\partial x}(F_1) = \frac{\partial}{\partial x}(xy) = y$$

$$\frac{\partial}{\partial y}(F_2) = \frac{\partial}{\partial y}(x^2+x) = 0$$

$$\operatorname{div}(F) = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 = y + 0 = y$$

$$\int_{-1}^{1}\int_{x^4+1}^{3-x^2} y \, dy \, dx = \int_{-1}^{1} \frac{1}{2}\left[y^2 \right|_{x^4+1}^{3-x^2} dx$$

$$= \frac{1}{2}\int_{-1}^{1} (3-x^2)^2 - (x^4+1)^2 \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} 9 + x^4 - 6x^2 - [x^8 + 1 + 2x^4] \, dx$$

$$= \cancel{\frac{1}{2}}\left[9x + \frac{1}{5}x^5 - 2x^3 - \frac{1}{9}x^9 - x - \frac{2}{5}x^5 \right|_{\cancel{-1}}^{1}$$

-1 to 1 same as $2x$

$$= \left(9 + \frac{1}{5} - 2 - \frac{1}{9} - 1 - \frac{2}{5}\right)$$

$$= 6 - \frac{1}{5} - \frac{1}{9} = \boxed{\frac{256}{45}}$$

Exercise 4 NAVIER-STOKES EQUATION.

The Navier-Stokes equation is the fundamental equation of fluid dynamics. In one of its many forms (incompressible and viscous flow) the equation is $\rho\left(\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}\right)=-\nabla p+\mu(\nabla\cdot\nabla)\mathbf{V}$. In the notation, $\mathbf{V}=<u,v,w>$ is the three-dimensional velocity field, p is the (scalar) pressure, ρ is the constant density of the fluid, and μ is the constant viscosity.

(i) Take the dot product of $\mathbf{V}$ and the nabla ∇ operator, then apply the result to u to show that $(\mathbf{V}\cdot\nabla)u=\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)u=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$.

(ii) Assume $u=xy^2z^3$ and find $(\mathbf{V}\cdot\nabla)u$ at $(1,1,1)$ where $\mathbf{V}=<1,x,1>$.

(iii) Write out the 1^{st} component equation of the Navier-Stokes vector equation.

(i) $V = \langle u, v, w \rangle \qquad \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$V \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

$(V\cdot\nabla)u = (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})u$

by distributive prop:

$= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ ✓

Also by:

$(a\cdot b)c = a\cdot(bc)$ associative prop. of dot w/scalar

$(V\cdot\nabla)u = V\cdot(\nabla u)$

$= \langle u, v, w\rangle \cdot \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\rangle$

$= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ ✓

(ii) $\nabla u = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$

$V\cdot\nabla u = y^2z^3 + 2x^2yz^3 + 3xy^2z^2$ @ $(x,y,z)=(1,1,1)$

$= \boxed{6}$

(iii) $\boxed{\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2}{\partial x^2}u}$