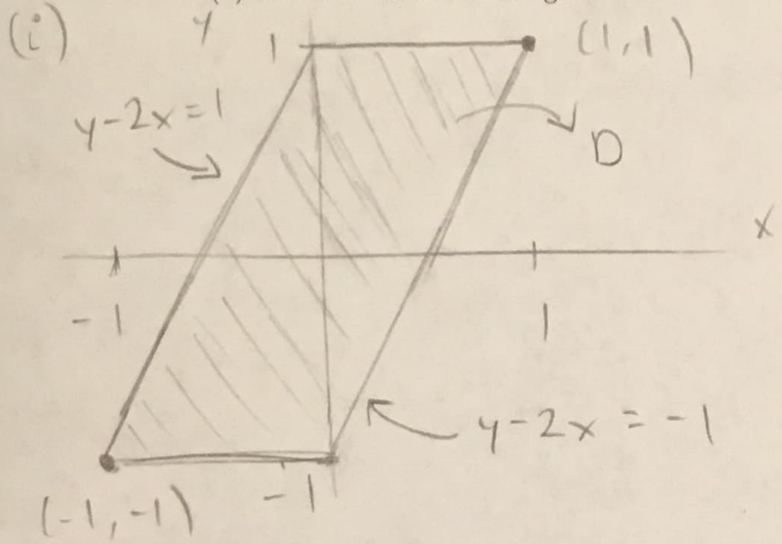
Exercise 1 CHANGE OF VARIABLES.

Let $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 ; |x| + |y - x| \leq 1\}$ and $\mathcal{I} = \iint_{\mathcal{D}} \frac{(y-2x)^2}{y^2+1} dxdy$.

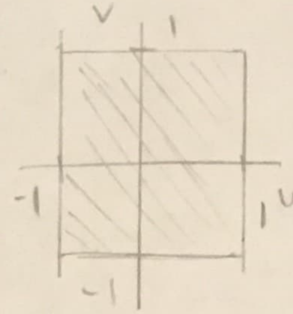
(i) Sketch the graph of the domain $\mathcal{D}$ and then use it to give a change of variables u and v.

(ii) Evaluate the double integral $\mathcal{I}$.

(i)



$$\boxed{\begin{cases} u = y - 2x \\ v = y \end{cases} \quad \begin{array}{l} -1 \leq u \leq 1 \\ -1 \leq v \leq 1 \end{array}}$$



$u = v - 2x$

$$\begin{cases} x = \frac{v-u}{2} \\ y = v \end{cases}$$

(ii) $\text{Jac}(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = -\frac{1}{2}$

$$\iint_D \frac{(y-2x)^2}{y^2+1} dxdy = \frac{1}{2} \int_{-1}^{1}\int_{-1}^{1} \frac{u^2}{v^2+1} du dv$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\frac{u^3}{3(v^2+1)} \Big|_{-1}^{1} \right) dv = \frac{1}{2} \int_{-1}^{1} \frac{1}{3(v^2+1)} + \frac{1}{3(v^2+1)} dv$$

$$= \frac{1}{3} \int_{-1}^{1} \frac{1}{v^2+1} dv = \frac{1}{3} \arctan(v) \Big|_{-1}^{1} = \frac{1}{3}\arctan(1) - \frac{1}{3}\arctan(-1)$$

$$= \frac{1}{3} \cdot \frac{\pi}{4} + \frac{1}{3} \cdot \frac{\pi}{4}$$

$$= \boxed{\frac{\pi}{6}}$$

Exercise 2 CONSERVATIVE VECTOR FIELD.

The vector field $\mathbf{F}(x,y) = \left\langle \frac{-x}{(x^2+y^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2)^{\frac{3}{2}}} \right\rangle$ is defined on the region $\mathcal{D} = \{(x,y) \neq (0,0)\}$.

(i) Is $\mathcal{D}$ a simply connected region?

(ii) Show that $\mathbf{F}$ satisfies the cross-partials condition. Does this guarantee that $\mathbf{F}$ is conservative?

(iii) Show that $\mathbf{F}$ is conservative on $\mathcal{D}$ by finding a potential function.

(i) D is not simply connected, since the unit circle cannot be deformed and contracted to a point in D because D doesn't include the origin.

(ii)
$$\frac{\partial F_1}{\partial y} = \frac{-x(2y)}{(x^2+y^2)^{\frac{5}{2}}} \cdot \left(-\frac{3}{2}\right) = \frac{3xy}{(x^2+y^2)^{5/2}}$$

$$\frac{\partial F_2}{\partial x} = \frac{-y(2x)}{(x^2+y^2)^{\frac{5}{2}}} \cdot \left(-\frac{3}{2}\right) = \frac{3xy}{(x^2+y^2)^{5/2}}$$

$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, F satisfies cross-partials condition

This doesn't guarantee that F is conservative since F's region D is not simply connected and a simply connected region is necessary for applying the relevant theorem.

(iii) $\vec{F} = \nabla\phi$

$$\nabla\phi = \int \frac{-x}{(x^2+y^2)^{3/2}}\,dx \qquad u = x^2+y^2 \quad du = 2x\,dx$$

$$\nabla\phi = \int -\frac{1}{2} u^{-3/2}\,du$$

$$= -\frac{1}{2}\cdot(-2)\,u^{-1/2} + c(y)$$

$$= (x^2+y^2)^{-1/2} + c_1(y)$$

$$\phi_y = \frac{-y}{(x^2+y^2)^{3/2}} = \frac{-\frac{1}{2}(2y)}{(x^2+y^2)^{3/2}} + c_1'(y)$$

$$0 = c_1'(y)$$

$\boxed{\phi = (x^2+y^2)^{-1/2}}$ is a potential function for F

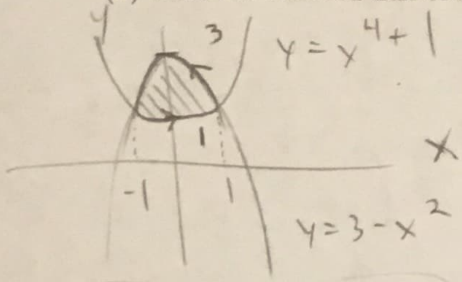
Exercise 3 GREEN's THEOREM.

Consider the region $\mathcal{D}$ bounded above by the curve $y = 3 - x^2$ and below by the curve $y = x^4 + 1$ and the vector field $\mathbf{F} = \left\langle xy, x^2 + x \right\rangle$. Using Green's Theorem

(i) Find the counterclockwise circulation of $\mathbf{F}$ around the boundary of $\mathcal{D}$.

(ii) Find the outward flux of $\mathbf{F}$ accross the boundary of $\mathcal{D}$.

(i)



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F})_z \, dA$$

$$D = \begin{cases} x^4 + 1 \le y \le 3 - x^2 \\ -1 \le x \le 1 \end{cases}$$

$$x^4 + 1 = 3 - x^2$$
$$x^4 + x^2 - 2 = 0$$
$$(x^2 + 2)(x^2 - 1) = 0$$
$$x = \pm 1$$

$$\nabla \times F = 2x + 1 - x = x + 1$$

$$\int_{-1}^{1}\int_{x^4+1}^{3-x^2} x + 1 \, dy\, dx$$

$$= \int_{-1}^{1} \left((x+1) y \Big|_{x^4+1}^{3-x^2} \right) dx$$

$$= \int_{-1}^{1} (x+1)(3 - x^2 - x^4 - 1)\, dx$$

$$= \int_{-1}^{1} (x+1)(2 - x^2 - x^4)\, dx = \int_{-1}^{1} 2x + 2 - x^3 - x^2 - x^5 - x^4 \, dx$$

$$= x^2 + 2x - \frac{x^4}{4} - \frac{x^3}{3} - \frac{x^6}{6} - \frac{x^5}{5} \Big|_{-1}^{1} = \left(1 + 2 - \frac{1}{4} - \frac{1}{3} - \frac{1}{6} - \frac{1}{5}\right)$$
$$- \left(1 - 2 - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \frac{1}{5}\right)$$
$$= 4 - \frac{2}{3} - \frac{2}{5} = \boxed{\frac{44}{15}}$$

(ii) $\iint_D \nabla \cdot F \, dA$

$$= \int_{-1}^{1}\int_{x^4+1}^{3-x^2} y \, dy\, dx = \int_{-1}^{1} \frac{y^2}{2} \Big|_{x^4+1}^{3-x^2} dx$$

$$= \int_{-1}^{1} \frac{(3-x^2)^2}{2} - \frac{(x^4+1)^2}{2} dx = \frac{1}{2}\int_{-1}^{1} -x^8 - x^4 - 6x^2 + 8 \, dx$$

$$= \frac{1}{2}\left(-\frac{x^9}{9} - \frac{x^5}{5} - 2x^3 + 8x \Big|_{-1}^{1}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{9} - \frac{1}{5} - 2 + 8 - \left(\frac{1}{9} + \frac{1}{5} + 2 - 8\right)\right) = -\frac{1}{9} - \frac{1}{5} - 2 + 8 = \boxed{\frac{256}{45}}$$

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Exercise 4 NAVIER-STOKES EQUATION.

The Navier-Stokes equation is the fundamental equation of fluid dynamics. In one of its many forms (incompressible and viscous flow) the equation is $\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right) = -\nabla p + \mu(\nabla\cdot\nabla)\mathbf{V}$. In the notation, $\mathbf{V} =< u, v, w >$ is the three-dimensional velocity field, p is the (scalar) pressure, ρ is the constant density of the fluid, and μ is the constant viscosity.

(i) Take the dot product of $\mathbf{V}$ and the nabla ∇ operator, then apply the result to u to show that $(\mathbf{V}\cdot\nabla)u = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$.

(ii) Assume $u = xy^2z^3$ and find $(\mathbf{V}\cdot\nabla)u$ at $(1,1,1)$ where $\mathbf{V} =< 1, x, 1 >$.

(iii) Write out the 1^{st} component equation of the Navier-Stokes vector equation.

(i) $V\cdot\nabla = \langle u, v, w\rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle$

$= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)$

$(V\cdot\nabla)u = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$

(ii) $(V\cdot\nabla)u = V_1\frac{\partial u}{\partial x} + V_2\frac{\partial u}{\partial y} + V_3\frac{\partial u}{\partial z}$ $\quad V = \langle V_1, V_2, V_3\rangle$

$= 1(y^2z^3) + x(2xyz^3) + 1(3xy^2z^2)$

$(x,y,z) = (1,1,1)$

$\Rightarrow 1(1) + 1(2(1)) + 1(3)(1) = 1+2+3$

$= \boxed{6}$

(iii) $\rho\left(\frac{\partial V_1}{\partial t} + (V\cdot\nabla)V_1\right) = -\frac{\partial p}{\partial x} + \mu(\nabla\cdot\nabla)V_1$

$\rho\left(\frac{\partial V_1}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} +$

$\mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u$

$$\boxed{\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$