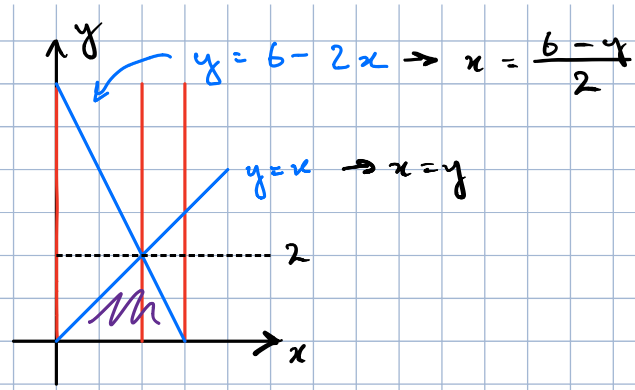
Exercise 1 DOUBLE INTEGRAL IN CARTESIAN SYSTEM OF COORDINATES.

Consider the integral $\int_0^2 \int_0^x dydx + \int_2^3 \int_0^{6-2x} dydx$.

(i) Reverse the order of integration to combine the sum above into one double integral.

(ii) Evaluate the integral.

(i)



$0 \le y \le 2$

$y \le x \le \frac{6-y}{2}$

$$\int_0^2 \int_0^x dy\,dx + \int_2^3 \int_0^{6-2x} dy\,dx = \boxed{\int_0^2 \int_y^{\frac{6-y}{2}} dx\,dy}$$

(ii)

$$\int_0^2 \int_y^{\frac{6-y}{2}} dx\,dy = \int_0^2 \frac{6-y}{2} - y\,dy = \int_0^2 3 - \frac{3y}{2}\,dy$$

$$= \left[3y - \frac{3y^2}{4}\right|_0^2$$

$$= 3\cdot 2 - \frac{3\cdot \cancel{4}}{\cancel{4}} - 0$$

$$= \boxed{3}$$

Exercise 2 DOUBLE INTEGRAL IN POLAR COORDINATES.

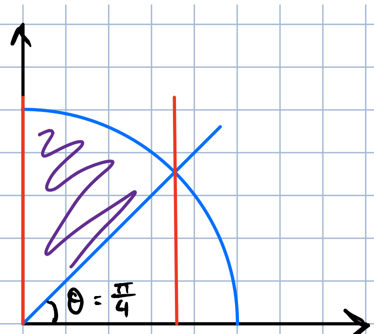
Consider the intergal $\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}} dydx + \int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{|y|}^{\sqrt{1-y^2}} dxdy$

(i) Use the polar coordinates to combine the integrals into a single double integral.

(ii) Evaluate the integral. (Assume $-\pi \le \theta \le \pi$)

(i) $0 \le x \le \frac{\sqrt{2}}{2}$

$x \le y \le \sqrt{1-x^2}$



lower y: $y = x$
$r\sin\theta = r\cos\theta$
$\tan\theta = 1$
$\theta = \frac{\pi}{4}$

lower x: $x = 0$
$r\cos\theta = 0$
$\theta = \frac{\pi}{2}$

$\rightarrow \frac{\pi}{2} \le \theta \le \frac{\pi}{4}$

$0 \le r \le 1$

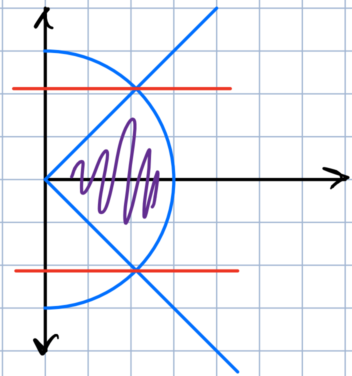
graph confirms

upper y: $y = \sqrt{1-x^2}$
$y^2 = 1 - x^2$
$x^2 + y^2 = 1$
$r^2 = 1$
$r = 1$

$$\int_0^{2\sqrt{2}}\int_x^{\sqrt{1-x^2}} dy\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^1 r\, dr\, d\theta$$

second integral conversion on next page.

$$-\frac{\sqrt{2}}{2} \le y \le \frac{\sqrt{2}}{2}$$



$$|y| \le x \le \sqrt{1-y^2}$$

$$-\frac{\sqrt{2}}{2} \le r\sin\theta \le \frac{\sqrt{2}}{2}$$

$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

$$0 \le r \le 1$$

$$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{|y|}^{\sqrt{1-y^2}} dy\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r\, dr\, d\theta$$

sum: $\displaystyle \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r\, dr\, d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r\, dr\, d\theta = \boxed{\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r\, dr\, d\theta}$

(ii) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r\, dr\, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} \left[r^2 \right|_0^1 d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\, d\theta$$

$$= \left[\frac{1}{2}\theta \right|_{-\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} + \frac{\pi}{8} = \boxed{\frac{3\pi}{8}}$$

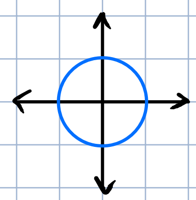
Exercise 3 TRIPLE INTEGRAL IN CYLINDRICAL COORDINATES.

Consider the region $\mathcal{W}$ that lies between the sphere $x^2+y^2+z^2=4$, above the plane $z=0$, and inside the cylinder $x^2+y^2=1$.

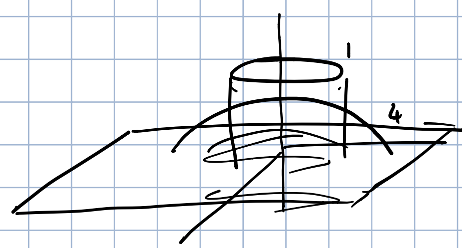
(i) Sketch the region $\mathcal{W}$.

(ii) Use cylindrical coordinates to integrate $f(x,y,z)=z$ over $\mathcal{W}$.

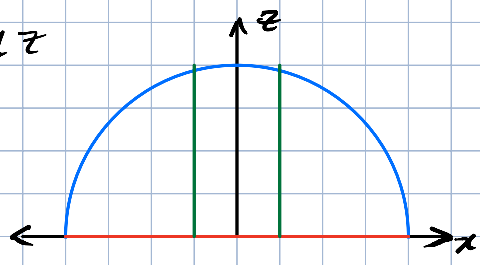
(i) polar:



3D:



xz

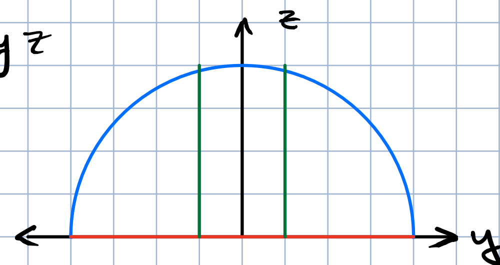


$x = \pm 1$

$z \le \sqrt{4-x^2}$

$z \ge 0$

yz



$y = \pm 1$

$z \le \sqrt{4-y^2}$

$z \ge 0$

(ii) $x^2+y^2+z^2 \le 4$

$z^2 \le 4-(x^2+y^2)$

$z^2 \le \sqrt{4-(x^2+y^2)} = \sqrt{4-r^2}$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$0 \le z \le \sqrt{4-r^2}$

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} z\cdot r \; dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}\left[z^2\right|_0^{\sqrt{4-r^2}} r\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r(4-r^2)\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 4r - r^3\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[2r^2 - \frac{1}{4}r^4\right|_0^1 d\theta = \frac{1}{2}\int_0^{2\pi}\frac{7}{4}\, d\theta = \frac{1}{2}\left[\frac{7}{4}\theta\right|_0^{2\pi}$$

$$= \boxed{\frac{7}{4}\pi}$$

Exercise 4 TRIPLE INTEGRAL IN SPHERICAL COORDINATES.

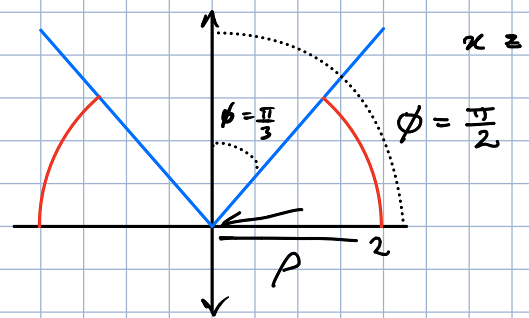
Consider the solid $\mathcal{W}$ bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$.

(i) Find the spherical coordinate limits for the integral that calculates the volume of the region $\mathcal{W}$.

(ii) Evaluate the integral.

(i)

$$\boxed{\begin{aligned} \frac{\pi}{3} &\le \phi \le \frac{\pi}{2} \\ 0 &\le \rho \le 2 \\ 0 &\le \theta \le 2\pi \end{aligned}}$$



xy is circle in $\mathbb{R}^2$, $0 \le \theta \le 2\pi$

(ii)

$$\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\frac{1}{3}\rho^3\right|_0^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{8}{3}\sin\phi \, d\phi \, d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} \left[-\cos\phi\right|_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta = \frac{8}{3}\int_0^{2\pi} 0 - \left(-\frac{1}{2}\right) d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} \frac{1}{2} \, d\theta = \frac{8}{3}\left[\frac{1}{\cancel{2}}\theta\right|_0^{\cancel{2}\pi}$$

$$= \boxed{\frac{8}{3}\pi}$$

Exercise 5 TRIPLE INTEGRAL IN CARTESIAN COORDINATES

Cnnsider the integral $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$.

(i) Rewrite the integral as an equivalent integral in the order $\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x,y,z)\,dy\,dx\,dz$.

(ii) Explain how you got the new limits of integration..

(i) $-1 \le x \le 1$ $\qquad x^2 \le y \le 1$ $\qquad 0 \le z \le 1-y$

$x \in [-1, 1] \longrightarrow y \in [0, 1] \longrightarrow y = 0$

$0 \le z \le 1$

$y \ge x^2$

$-\sqrt{y} \le x \le \sqrt{y}$

$z \le 1-y$

$y \le 1-z$

$-\sqrt{1-z} \le x \le \sqrt{1-z}$

$x^2 \le y \le 1-z$

$$\boxed{\int_{0}^{1}\int_{\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} f(x,y,z)\,dy\,dx\,dz}$$

(ii) Explanation: first, I wrote the integral bounds as inequalities. Since we want z to be on the outside, I found its bounds using the bounds of x, and resulting bounds of y. I then solved for x using the y inequality, and substituted y with y in terms of z. Finally I found the range of y using the existing inequalities.