Exercise 1 DOUBLE INTEGRAL IN CARTESIAN SYSTEM OF COORDINATES.

Consider the integral $\int_0^2 \int_0^x dydx + \int_2^3 \int_0^{6-2x} dydx.$

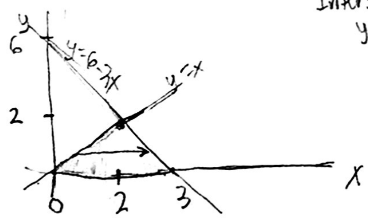
(i) Reverse the order of integration to combine the sum above into one double integral.

(ii) Evaluate the integral.

i)

$0 \le y \le x$ $\quad 0 \le y \le 6-2x$

$0 \le x \le 2$ $\quad 2 \le x \le 3$



Intersection

$y=x \quad y=6-2x$

$x=6-2x$

$3x=6$

$x=2$

convert to horizontally simple

$y=6-2x$ $\quad x=y$

$y-6=-2x$

$\frac{y-6}{-2}=x$

$x=-\frac{1}{2}y+3$

$y \le x \le -\frac{1}{2}y+3$

$0 \le y \le 2$

$f(x,y)=1$

$\boxed{\int_0^2 \int_y^{-\frac{1}{2}y+3} dxdy}$

ii) $\int_0^2 \int_y^{-\frac{1}{2}y+3} dxdy$

$\int_y^{-\frac{1}{2}y+3} dx = x \Big|_y^{-\frac{1}{2}y+3}$ $\quad -\frac{1}{2}y+3-y = -\frac{3}{2}y+3$ $\qquad \int_0^2 \int_y^{-\frac{1}{2}y+3} dxdy =$

$\int_0^2 -\frac{3}{2}y+3\,dy = -\frac{3}{4}y^2+3y \Big|_0^2$ $\quad \left(-\frac{3}{4}(2^2)+3(2)\right)-(0+0) = -3+6 = \boxed{3}$

Exercise 2 DOUBLE INTEGRAL IN POLAR COORDINATES.

Consider the intergal $\int_0^{\sqrt{2}/2} \int_x^{\sqrt{1-x^2}} dydx + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{|y|}^{\sqrt{1-y^2}} dxdy$

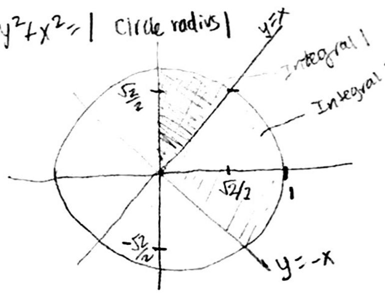
(i) Use the polar coordinates to combine the integrals into a single double integral.

(ii) Evaluate the integral. (Assume $-\pi \leq \theta \leq \pi$)

i) $x \leq y \leq \sqrt{1-x^2}$ $|y| \leq x \leq \sqrt{1-y^2}$

$0 \leq x \leq \frac{\sqrt{2}}{2}$ $-\frac{\sqrt{2}}{2} \leq y \leq \frac{\sqrt{2}}{2}$

$y^2+x^2=1$ circle radius 1



$x=|y|$

$x=\pm y$

Boundaries in polar:

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ $\theta = \tan^{-1}\left(\frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$0 \leq r \leq 1$ circle radius 1

$f(x,y)=1$ $f(r,\theta)=1$

$$\boxed{\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r\,dr\,d\theta}$$

ii) $\int_{-\frac{\pi}{4}}^{\pi/2} \int_0^1 r\,dr\,d\theta$

$\int_0^1 r\,dr = \frac{1}{2}r^2\Big|_0^1 \quad \frac{1}{2} - 0 = \frac{1}{2}$

$\int_{-\pi/4}^{\pi/2} \frac{1}{2} d\theta = \frac{1}{2}\theta \Big|_{-\frac{\pi}{4}}^{\pi/2} \quad \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{4}\right) = \frac{\pi}{4} + \frac{\pi}{8} = \frac{2\pi}{8} + \frac{\pi}{8} = \boxed{\frac{3\pi}{8}}$

$\int_{-\frac{\pi}{4}}^{\pi/2} \int_0^1 r\,dr\,d\theta =$

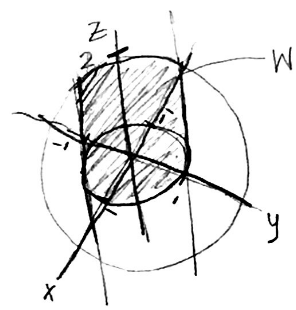
Exercise 3 TRIPLE INTEGRAL IN CYLINDRICAL COORDINATES.

Consider the region $\mathcal{W}$ that lies between the sphere $x^2+y^2+z^2=4$, above the plane $z=0$, and inside the cylinder $x^2+y^2=1$.

(i) Sketch the region $\mathcal{W}$.

(ii) Use cylindrical coordinates to integrate $f(x,y,z)=z$ over $\mathcal{W}$.

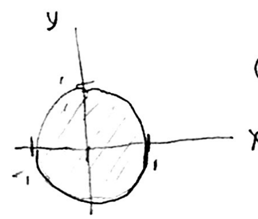
i)



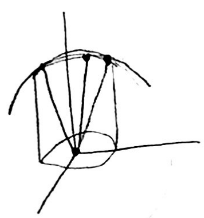
Sphere : $x^2+y^2+z^2=4$
centered origin
radius = 2

Cylinder : $x^2+y^2=1$
Project xy plane = circle radius 1

ii) W:



$0 \le \theta \le 2\pi$
$0 \le r \le 1$



$0 \le z \le$ sphere

$x^2+y^2+z^2=4$
$r^2+z^2=4$
$z=\pm\sqrt{4-r^2}$ positive since above plane $z=0$
$z=\sqrt{4-r^2}$

$0 \le z \le \sqrt{4-r^2}$

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} z\,r\,dz\,dr\,d\theta$$

$f(x,y,z)=z$
in cylindrical:
$f(r,y,z)=z$

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} z\,r\,dz\,dr\,d\theta$$

$$\int_0^{\sqrt{4-r^2}} zr\,dz = \frac{1}{2}z^2 r\Big|_0^{\sqrt{4-r^2}} \quad \frac{1}{2}(\sqrt{4-r^2})^2 r - 0 = \frac{1}{2}(4-r^2)r = \frac{1}{2}(4r-r^3) = 2r-\frac{1}{2}r^3$$

$$\int_0^1 2r-\frac{1}{2}r^3\,dr = r^2-\frac{1}{8}r^4\Big|_0^1 \quad 1-\frac{1}{8}-(0-0) = \frac{8}{8}-\frac{1}{8} = \frac{7}{8}$$

$$\int_0^{2\pi}\frac{7}{8}\,d\theta = \frac{7}{8}\theta\Big|_0^{2\pi} \quad \frac{7}{8}(2\pi)-0 = \boxed{\frac{7\pi}{4}} = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} z\,r\,dz\,dr\,d\theta$$

Exercise 4 TRIPLE INTEGRAL IN SPHERICAL COORDINATES.

Consider the solid $\mathcal{W}$ bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$.

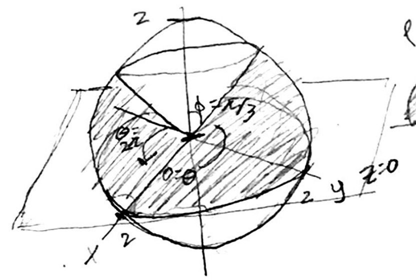
(i) Find the spherical coordinate limits for the integral that calculates the volume of the region $\mathcal{W}$.

(ii) Evaluate the integral.

Volume $= \iiint_W dV$

$$\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

i)



$\rho = z$ so sphere radius 2

Intersection with $z=0$

$x^2+y^2=4$

circle radius 2

Spherical Boundaries:

$\frac{\pi}{3} \le \phi \le \frac{\pi}{2}$

$0 \le \theta \le 2\pi$

$0 \le \rho \le 2$

ii) $\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$\int_0^2 \rho^2 \sin\phi \, d\rho = \frac{1}{3}\rho^3 \sin\phi \Big|_0^2 \quad \frac{1}{3}2^3 \sin\phi - 0 = \frac{8}{3}\sin\phi$

$\int_{\pi/3}^{\pi/2} \frac{8}{3}\sin\phi \, d\phi = -\frac{8}{3}\cos\phi \Big|_{\pi/3}^{\pi/2} \quad \cancel{-\frac{8}{3}\cos\left(\frac{\pi}{2}\right)} + \frac{8}{3}\cos\left(\frac{\pi}{3}\right) = \frac{8}{3}\left(\frac{1}{2}\right) = \frac{8}{6} = \frac{4}{3}$

$\int_0^{2\pi} \frac{4}{3} d\theta = \frac{4}{3}\theta \Big|_0^{2\pi} \quad \frac{4}{3}(2\pi) - \cancel{\frac{4}{3}(0)} = \boxed{\frac{8\pi}{3}}$

$\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$

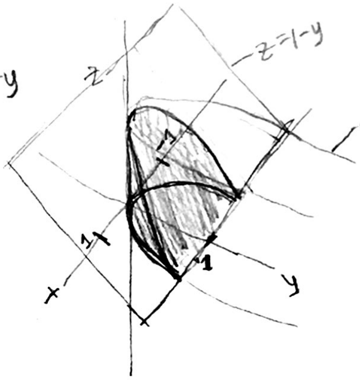
Exercise 5 TRIPLE INTEGRAL IN CARTESIAN COORDINATES

Cnnsider the integral $\displaystyle\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx.$

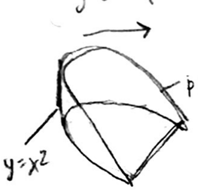
(i) Rewrite the integral as an equivalent integral in the order $\displaystyle\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x,y,z)\,dy\,dx\,dz.$

(ii) Explain how you got the new limits of integration..

i) $0 \le z \le 1-y$
$x^2 \le y \le 1$
$-1 \le x \le 1$

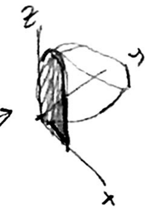


y-simple



$x^2 \le y \le 1-z$

$z=1-y$ $\quad z+y=1$
$y=1-z$



Projection xy-plane for "shadow"

horizontally simple (dxdz)

$-\sqrt{1-z} \le x \le \sqrt{1-z}$
$0 \le z \le 1$

Intersection of
$z=1-y$ and $y=x^2$
$z=1-x^2$
$z+x^2=1$
$x^2=1-z$
$x=\pm\sqrt{1-z}$

Boundaries: $x^2 \le y \le 1-z$
$-\sqrt{1-z} \le x \le \sqrt{1-z}$
$0 \le z \le 1$

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx = \boxed{\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} f(x,y,z)\,dy\,dx\,dz}$$

ii) I first drew the region. dydxdz order of integration means the region is y-simple, so I found the two surfaces the volume was bounded by in the y-direction, which was the $y=x^2$ parabola extended for all z, and the plane $z=1-y$, I rewrote the two surfaces in terms $f(x,z)$ so $x^2 \le y \le 1-z$.

Next I found the projection of the volume in the xy-plane (the shadow of the object) to find the domain for x and z. The intersection is the projected shadow so I projected the intersection of $z=1-y$ and $y=x^2$ which is $x=\pm\sqrt{1-z}$ (written in $f(z)$ since horizontally simple domain integrate in order dxdz), z was between 0 and 1 so the bounds for integrating in order dxdz were $-\sqrt{1-z} \le x \le \sqrt{1-z}$ and $0 \le z \le 1$.