Final Exam

MATH 32B @ UCLA (WINTER 2021)

Assigned: March 16, 2021.

Instructions/Admonishment

1. SHOW ALL WORK.
   A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit.
2. Duration: 24 hours.
3. The following is my own work, without the aid of any other person.
   Signature: ____

Exercise 1 DOUBLE INTEGRAL.

(i) Prove the formula $\int_0^1 \int_0^y f(x)dxdy = \int_0^1 (1-x)f(x)dx$.

(ii) Use (i) to evaluate $\int_0^1 \int_0^y \frac{\sin x}{1-x}dxdy$.

Exercise 2 TRIPLE INTEGRAL IN CARTESIAN COORDINATES

Consider the integral $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)dydzdx$.

(i) Rewrite the integral as an equivalent integral in the order $\int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)dxdydz$.

(ii) Explain how you got the new limits of integration.

Exercise 3 CHANGE OF VARIABLES.

Let $\mathscr{R}$ be the rectangle enclosed by the lines $y = x$, $y = x - 2$, $x + y = 0$, and $x + y = 3$.

(i) Give a change of variables u and v and find the image of the rectangle $\mathscr{R}$ in the uv-plane using this transformation.

(ii) Compute $\iint_{\mathscr{R}} (x + y)e^{x^2 - y^2}\, dA$.

Exercise 4 CONSERVATIVE VECTOR FIELD

Consider the vector field $\mathbf{F}(x,y,z) = \left\langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \right\rangle$.

(i) Show that $\mathbf{F}$ is a conservative vector field and find ϕ such that $\mathbf{F} = \nabla\phi$.

(ii) Use this to evaluate $\displaystyle\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathscr{C}$ consists of the line segment from $(1,0,-2)$ to $(1,1,0)$ followed by the curve given given by $\mathbf{r}(t) = \langle e^t, \cos(t), t\rangle$ for $0 \leq t \leq 1$.

Exercise 5 GREEN's THEOREM.

Consider the line integral $\mathscr{I} = \oint_{\mathscr{C}} xydy - y^2dx$ where $\mathscr{C}$ is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

(i) Find the line integral $\mathscr{I}$ using the circulation form of Green's Theorem.

(ii) Find the line integral $\mathscr{I}$ using the flux form of Green's Theorem.

Exercise 6 STOKES' THEOREM.

Use Stokes' Theorem to find the circulation of the vector field $\mathbf{F} = \langle x^2 - y, 4z, x^2 \rangle$ around the curve $\mathscr{C}$, given by the intersection of the plane z=2 and the cone $z = \sqrt{x^2 + y^2}$, counterclockwise oriented as viewed from above.

Exercise 7 DIVERGENCE THEOREM.

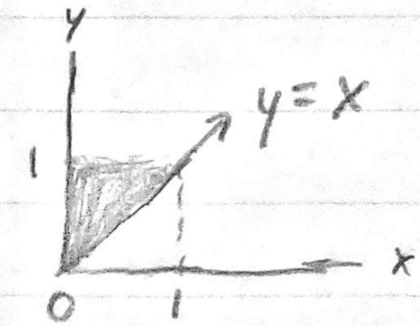
Let $\mathbf{F}(x, y, z) = \left\langle 3xy^2, xe^z, z^3 + 4x \right\rangle$ and $\mathscr{S}$ be a surface of the solid bounded by the cylinder $y^2+z^2 = 1$ and the planes $x = -1$ and $x = 2$.

(i) Sketch the surface $\mathscr{S}$.

(ii) Use Divergence Theorem to compute $\displaystyle\int\int_{\mathscr{S}} \mathbf{F} \cdot d\mathbf{S}$

Exercise 1

i) $\int_0^1 \int_0^y f(x)\, dx\, dy$ bounded by



$x \leq y \leq 1$

$0 \leq x \leq 1$

$$= \int_0^1 \int_x^1 f(x)\, dy\, dx$$

$$= \int_0^1 f(x)\, x \Big|_x^1 dx$$

$$= \boxed{\int_0^1 (1-x)\, f(x)\, dx}$$

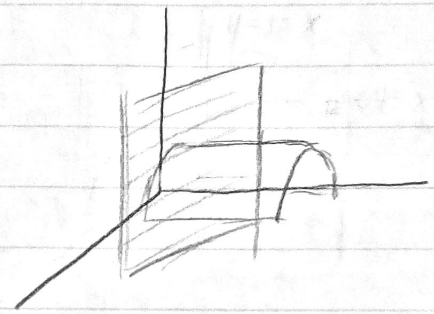
ii) $$\int_0^1 \int_0^y \frac{\sin x}{1-x}\, dx\, dy = \int_0^1 (1-x) \frac{\sin x}{1-x}\, dx$$

$$= \int_0^1 \sin x\, dx$$

$$= -\cos x \Big|_0^1 = \boxed{-(\cos(1) - 1)}$$

Exercise 2

i) $0 \leq x \leq 1$
$0 \leq z \leq 1-x^2$
$0 \leq y \leq 1-x$

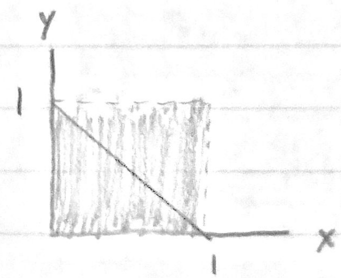


$y = 1-x \rightarrow x = 1-y$
$z = 1-x^2 = 1-(1-y)^2$
$= 1-y^2+2y-1 = -y^2+2y$

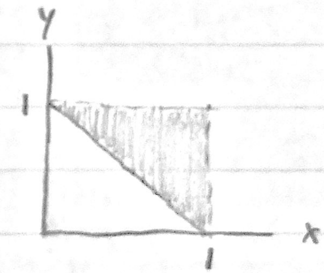
$x^2 = 1-z \rightarrow x = \sqrt{1-z}$
$y = 1-x = 1-\sqrt{1-z}$

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz - \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_{1-y}^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz$$

ii) I got the new limits of integration by taking the region bounded by $z=1-x^2$ and the unit cube, and subtracting the region bounded by $z=1-x^2$, the unit cube, and $y=1-x$ (the region in the positive y-direction of $y=1-x$. This results in the region bounded by $0 \leq x \leq 1$, $0 \leq z \leq 1-x^2$, and $0 \leq y \leq 1-x$.



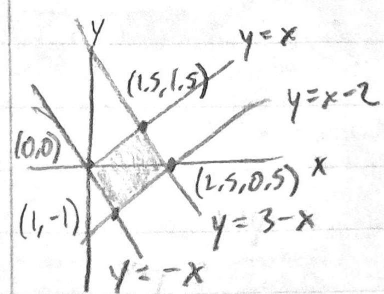
minus



top-down view

Exercise 3

i)



let $\boxed{G(u,v) = (x+y,\ x-y)}$

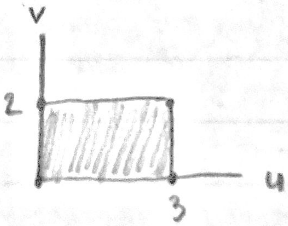
rectangle represents rotated cartesian plane with $y=-x$ and $y=x$ as axes

On u-v plane:

$(0,0) \longrightarrow (0,0)$ $\qquad$ $(1.5,1.5) \longrightarrow (3,0)$

$(1,-1) \longrightarrow (0,2)$ $\qquad$ $(2.5,0.5) \longrightarrow (3,2)$

so $0 \leq u \leq 3$ and $0 \leq v \leq 2$



ii)

$$\iint_R (x+y)\, e^{x^2-y^2}\, dA = \iint_R (x+y)\, e^{(x+y)(x-y)}\, dA$$

$$= \int_0^3 \int_0^2 u e^{uv}\, |Jac(G)|\, du\, dv$$

$$Jac(G) = \frac{1}{(1\cdot -1) - (1 \cdot 1)} = \frac{1}{-2}$$

$$|Jac(G)| = \frac{1}{2}$$

$$= \int_0^3 \frac{1}{2} e^{uv} \Big|_0^2\, du$$

$$= \int_0^3 \frac{1}{2}(e^{2u} - 1)\, du = \frac{1}{4}e^{2u} - \frac{1}{2}u \Big|_0^3 = \frac{1}{4}e^6 - \frac{3}{2} - \frac{1}{4}$$

$$= \boxed{\frac{1}{4}(e^6 - 7)}$$

Exercise 4

i)

$$\frac{\partial F_1}{\partial y} = 6xy^2z^4 \qquad \frac{\partial F_2}{\partial x} = 6xy^2z^4 \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \ \checkmark$$

$$\frac{\partial F_2}{\partial z} = 12x^2y^2z^3 \qquad \frac{\partial F_3}{\partial y} = 12x^2y^2z^3 \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \ \checkmark$$

$$\frac{\partial F_3}{\partial x} = 8xy^3z^3 \qquad \frac{\partial F_1}{\partial z} = 8xy^3z^3 \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \ \checkmark$$

$$\phi_x(x,y,z) = 2xy^3z^4$$

$$\phi(x,y,z) = \int \phi_x(x,y,z)\,dx = x^2y^3z^4 + c(y,z)$$

$$\phi_y(x,y,z) = 3x^2y^2z^4 + c_y(y,z) \longrightarrow c_y(y,z) = 0 \quad \text{so } c(y,z) = \text{const}$$

$$\phi(x,y,z) = \int \phi_y(x,y,z)\,dy = x^2y^3z^4 + A(z)$$

$$\phi_z(x,y,z) = 4x^2y^3z^3 + A'(z) \longrightarrow A'(z) = 0 \quad \text{so } A(z) = \text{const}$$

$$\boxed{\phi(x,y,z) = x^2y^3z^4 + \text{const}}$$

ii) F is conservative so:

$$\int_C F \cdot dr = \phi(e, \cos(1), 1) - \phi(1, 0, -2)$$

$$= e^2\cos^3(1) \cdot 1^4 - 0$$

$$= \boxed{e^2\cos^3(1)}$$

same as end of first line segment so path is continuous

$$r(0) = (1, 1, 0)$$

$$r(1) = (e, \cos(1), 1)$$

Exercise 5

i) $\oint_C F_1 dx + F_2 dy = \iint_D \text{curl}_z(F)\, dA$

$F = \langle F_1, F_2 \rangle = \langle -y^2, xy \rangle$

$$\text{curl}(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ -y^2 & xy & 0 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \partial x & \partial y \\ -y^2 & xy \end{matrix} = \langle 0, 0, 3y \rangle$$

$$\int_0^1 \int_0^1 \text{curl}_z(F)\, dA = \int_0^1 \int_0^1 3y \, dx\, dy$$

$$= \int_0^1 3yx \Big|_0^1 dy$$

$$= \int_0^1 3y\, dy = \frac{3y^2}{2} \Big|_0^1 = \boxed{\frac{3}{2}}$$

ii) $\oint_C F_1 dy - F_2 dx = \iint_D \text{div}(F)\, dA$

$F = \langle F_1, F_2 \rangle = \langle xy, y^2 \rangle$

$$\text{div}(F) = \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y} y^2$$
$$= y + 2y$$
$$= 3y$$

$$\int_0^1 \int_0^1 \text{div}(F)\, dA = \int_0^1 \int_0^1 3y\, dx\, dy$$

$$= \int_0^1 3yx \Big|_0^1 dy$$

$$= \int_0^1 3y\, dy = \frac{3y^2}{2} \Big|_0^1 = \boxed{\frac{3}{2}}$$

Exercise 6

$G(r,\theta) = (r\cos\theta, r\sin\theta, r)$ $\quad 0 \le \theta \le 2\pi$

$\quad 0 \le r \le 2$

$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot \vec{n}\, ds$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ x^2-y & 4z & x^2 \end{vmatrix} \begin{matrix} i & j \\ \partial x & \partial y \\ x^2-y & 4z \end{matrix} = \langle -4, -2x, 1 \rangle$$

$N(r,\theta) = T_r \times T_\theta$

$T_r = \langle \cos\theta, \sin\theta, 1 \rangle$ $\quad T_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$N(r,\theta) = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \begin{matrix} i & j \\ \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{matrix} = \langle -r\cos\theta, -r\sin\theta, \cancel{r\cos^2\theta + r\sin^2\theta}\ r \rangle$$

$(\nabla \times F) \cdot N(r,\theta) = \langle -4, -2x, 1 \rangle \cdot \langle -r\cos\theta, -r\sin\theta, r \rangle$

$= \langle -4, 2r\cos\theta, 1 \rangle \cdot \langle -r\cos\theta, -r\sin\theta, r \rangle$

$= 4r\cos\theta - 2r^2\cos\theta\sin\theta + r$

$$\int_0^{2\pi}\int_0^2 (4r\cos\theta - 2r^2\cos\theta\sin\theta + r)\, dr\, d\theta$$

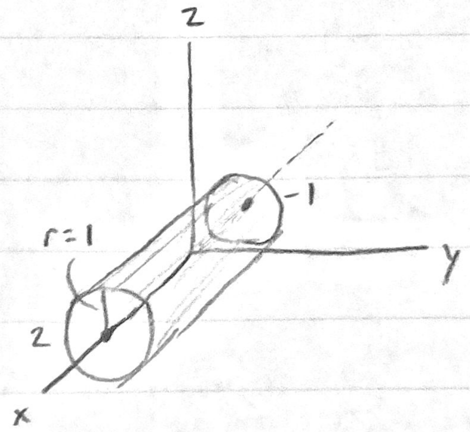
$$\int_0^{2\pi} \left(2r^2\cos\theta - \frac{2}{3}r^3\cos\theta\sin\theta + \frac{r^2}{2}\right)\Big|_0^2 d\theta$$

$$\int_0^{2\pi} \left(8\cos\theta - \frac{16}{3}\cos\theta\sin\theta + 2\right) d\theta$$

$$8\sin\theta - \frac{16}{3}\left(-\frac{\cos^2\theta}{2}\right) + 2\theta \Big|_0^{2\pi} = \frac{16}{3}\left(\cancel{\tfrac{1}{2} - \tfrac{1}{2}}\ 0\right) + 2(2\pi) = \boxed{4\pi}$$

Exercise 7

i)



ii) $\iint_S F \cdot ds = \iiint_W \text{div}(F)\, dV$

$$\text{div}(F) = \nabla \cdot F = \frac{\partial}{\partial x} 3xy^2 + \frac{\partial}{\partial y} xe^2 + \frac{\partial}{\partial z}(z^3 + 4x)$$
$$= 3y^2 + 0 + 3z^2$$
$$= 3y^2 + 3z^2$$

$$G(x, r, \theta) = (x, r\cos\theta, r\sin\theta)$$

$$\int_{-1}^{2}\int_{0}^{2\pi}\int_{0}^{1} (3y^2 + 3z^2)\, dA = \int_{-1}^{2}\int_{0}^{2\pi}\int_{0}^{1} 3(r^2\cos^2\theta + r^2\sin^2\theta)\, r\,dr\,d\theta\,dx$$
$$= \int_{-1}^{2}\int_{0}^{2\pi}\int_{0}^{1} 3r^3\, dr\,d\theta\,dx$$
$$= \int_{-1}^{2}\int_{0}^{2\pi} \frac{3}{4}\, d\theta\, dx$$
$$= \int_{-1}^{2} \frac{3}{4}\theta \Big|_0^{2\pi} dx$$
$$= \int_{-1}^{2} \frac{3}{2}\pi\, dx = \frac{3}{2}\pi x \Big|_{-1}^{2} = \boxed{\frac{9\pi}{2}}$$