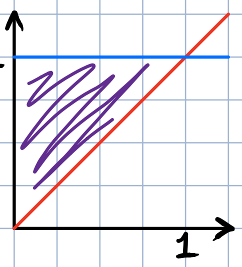
Exercise 1 DOUBLE INTEGRAL.

(i) Prove the formula $\int_0^1 \int_0^y f(x)dxdy = \int_0^1 (1-x)f(x)dx$.

(ii) Use (i) to evaluate $\int_0^1 \int_0^y \frac{\sin x}{1-x} dxdy$.

(i)



$$\left.\begin{array}{l} 0 \le x \le y \\ 0 \le y \le 1 \end{array}\right\} \longrightarrow \quad 0 \le x \le y \le 1$$

between: $x = y$ and $x = 0$

$y = 0$ and $y = 1$

Looking at graph/ inequality we see:

$y = x$ and $y = 1$

$x = 0$ and $x = 1$

New bounds: $\begin{cases} 0 \le x \le 1 \\ x \le y \le 1 \end{cases}$

$$\int_0^1 \int_0^y f(x)\,dx\,dy = \int_0^1 \int_x^1 f(x)\,dy\,dx = \int_0^1 f(x)\,dx \int_x^1 dy$$

$$= \int_0^1 (1-x) f(x)\,dx$$

✓

(ii) Let's set $f(x) = \frac{\sin x}{1-x}$

$$\int_0^1 \int_0^y \frac{\sin x}{1-x}\,dx\,dy = \int_0^1 \cancel{(1-x)} \frac{\sin x}{\cancel{1-x}}\,dx = \int_0^1 \sin x\,dx$$

$$= -\cos x \Big|_0^1$$

$$= \boxed{1 - \cos(1)}$$

Exercise 2 TRIPLE INTEGRAL IN CARTESIAN COORDINATES

Consider the integral $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dydzdx$.

(i) Rewrite the integral as an equivalent integral in the order $\int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)\,dxdydz$.

(ii) Explain how you got the new limits of integration.

(i) $0 \le z \le 1 - x^2$ $\qquad$ $0 \le y \le 1 - x$

$|x| \le \sqrt{1-z^2}$

$0 \le z \le 1$

for $x = 0$

$\Rightarrow 0 \le y \le 1 - \sqrt{1-z^2}$

— or —

$0 \le z \le 1$

$0 \le x \le 1 - y$

$z \le (1-x)(1+x)$

$z \le y(1 + 1 - y)$

$z \le 2y - y^2$

$1 - \sqrt{1-z} \le y \le 1 + \sqrt{1-z}$

for $z = 0$

$1 - \sqrt{1-z} \le y \le 1$

$$\boxed{\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1 \int_{1-\sqrt{1-z}}^{1} \int_0^{1-y} f(x,y,z)\,dx\,dy\,dz}$$

(ii) First, I thought of the problem visually. I realized that for y close to zero, $z = 1 - x^2$. But, for y close to 1, the boundary is the plane $y = 1 - x$. With this in mind, I found the inequalities with x bounded by the $z = 1 - x^2$ and $y = 1 - x$ and added them together. I found the bounds of z were always $[0, 1]$, as x could be zero for both sets of bounds. The bounds of y did change as them.

Exercise 3 CHANGE OF VARIABLES.

Let $\mathscr{R}$ be the rectangle enclosed by the lines $y = x$, $y = x - 2$, $x + y = 0$, and $x + y = 3$.

(i) Give a change of variables u and v and find the image of the rectangle $\mathscr{R}$ in the uv-plane using this transformation.

(ii) Compute $\displaystyle\iint_{\mathscr{R}} (x + y)e^{x^2-y^2}\, dA$.

(i)
$$\left.\begin{aligned} y - x &= 0 \\ y - x &= -2 \end{aligned}\right\} \quad -2 \le y - x \le 0$$

$$\left.\begin{aligned} x + y &= 0 \\ x + y &= 3 \end{aligned}\right\} \quad 0 \le x + y \le 3$$

$u = y - x$

$v = x + y \longrightarrow y = v - x$

$u = v - x - x$

$u = v - 2x$

$2x = v - u$

$x = \dfrac{v-u}{2}$

$$y = v - \frac{v-u}{2} = \frac{u+v}{2}$$

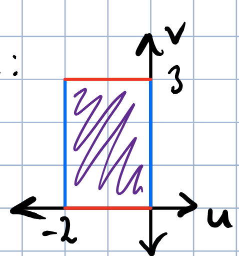
$$G(u, v) = \left(\frac{v-u}{2}, \frac{u+v}{2}\right) \qquad \mathrm{Jac}(G) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

for $-2 \le u \le 0$

$0 \le v \le 3$

$$\mathrm{Jac}(G) = -\frac{1}{2}$$

image:



(ii)
$$\iint_R (x+y)e^{x^2-y^2}\, dA = -\frac{1}{2}\int_0^3\int_{-2}^0 v e^{uv}\, du\, dv = -\frac{1}{2}\int_0^3 e^{uv}\Big|_{-2}^0$$

$$= -\frac{1}{2}\int_0^3 1 - e^{-2v}\, dv = -\frac{1}{2}\left[v + \frac{e^{-2v}}{2}\right|_0^3 = -\frac{1}{2}\left(3 + \frac{e^{-6}}{2} - \frac{1}{2}\right) = \boxed{\frac{-e^{-6} - 5}{4}}$$

Exercise 4 CONSERVATIVE VECTOR FIELD

Consider the vector field $\mathbf{F}(x,y,z) = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3\rangle$.

(i) Show that $\mathbf{F}$ is a conservative vector field and find ϕ such that $\mathbf{F} = \nabla\phi$.

(ii) Use this to evaluate $\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathscr{C}$ consists of the line segment from $(1,0,-2)$ to $(1,1,0)$ followed by the curve given given by $\mathbf{r}(t) = \langle e^t, \cos(t), t\rangle$ for $0 \le t \le 1$.

(i) Check cross-partials:

$\frac{\partial F_1}{\partial y} = 6xy^2z^4$ $\quad$ $\frac{\partial F_2}{\partial x} = 6xy^2z^4$ ✓ equal

$\frac{\partial F_2}{\partial z} = 12x^2y^2z^3$ $\quad$ $\frac{\partial F_3}{\partial y} = 12x^2y^2z^3$ ✓ equal

$\frac{\partial F_3}{\partial x} = 8xy^3z^3$ $\quad$ $\frac{\partial F_1}{\partial z} = 8xy^3z^3$ ✓ equal

$\boxed{\text{F must be conservative because the cross-partials condition is met}}$

$$\phi = \int F_1\,dx = \int 2xy^3z^4\,dx = x^2y^3z^4 + f(y,z)$$
$$\phi = \int F_2\,dy = \int 3x^2y^2z^4\,dy = x^2y^3z^4 + g(x,z)$$
$$\phi = \int F_3\,dy = \int 4x^2y^3z^3\,dz = x^2y^3z^4 + h(x,y)$$

$$x^2y^3z^4 + f(y,z) = x^2y^3z^4 + g(x,z) = x^2y^3z^4 + h(x,y)$$
$$f(y,z) = g(x,z) = h(x,y) = 0$$
$$\boxed{\phi(x,y,z) = x^2y^3z^4}$$

(ii)
$$\int_C F\cdot d\vec{r} = \phi(Q) - \phi(P) = \boxed{e^2\cos^3(1)}$$

$P = \text{start} = (1, 0, -2)$ $\quad$ $\phi(P) = 1^2\cdot 0^3\cdot -2^4 = 0$

$Q = \text{end} = (e^1, \cos(1), 1)$ $\quad$ $\phi(Q) = e^2\cos^3(1)\cdot 1^4$

Exercise 5 GREEN's THEOREM.

Consider the line integral $\mathscr{I} = \oint_{\mathscr{C}} xydy - y^2dx$ where $\mathscr{C}$ is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

(i) Find the line integral $\mathscr{I}$ using the circulation form of Green's Theorem.

(ii) Find the line integral $\mathscr{I}$ using the flux form of Green's Theorem.

(i) $$\oint_{\partial D} F \cdot dr = \iint_D \mathrm{curl}_z(F)\, dA$$

$$F = \langle -y^2, xy \rangle = \langle P, Q \rangle$$

$$\mathrm{curl}_z(F) = \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} = y - (-2y) = 3y$$

$0 \le x \le 1$

$0 \le y \le 1$

$$\int_0^1 \int_0^1 3y \, dy\, dx = \left[\frac{3}{2} y^2\right|_0^1 = \boxed{\frac{3}{2}}$$

(ii) Flux form:

for $F = \langle P, Q \rangle$ and unit normal N

$$\oint_C F \cdot N \, ds = \oint_C \mathrm{div}(F) dA = \iint_D P_x + Q_y \, dA$$

We can rewrite

$$\oint_C F \cdot N ds = \oint_C -Q\, dx + P\, dy$$

in our case: $Q = y^2$ $\quad Q_y = 2y$

$P = xy$ $\quad P_x = y$

$$\int_0^1 \int_0^1 P_x + Q_y \, dx\, dy = \int_0^1 \int_0^1 3y \, dx\, dy = \boxed{\frac{3}{2}}$$

Exercise 6 STOKES' THEOREM.

Use Stokes' Theorem to find the circulation of the vector field $\mathbf{F} = \langle x^2 - y, 4z, x^2 \rangle$ around the curve $\mathscr{C}$, given by the intersection of the plane z=2 and the cone $z = \sqrt{x^2 + y^2}$, counterclockwise oriented as viewed from above.

Boundary :

$$\sqrt{x^2 + y^2} = z = z = 2$$

$$x^2 + y^2 = 4 \quad @ \quad z = 2$$

Circle radius r, at height z=2

$$G(r, \theta) = \langle r\cos\theta, r\sin\theta, 2 \rangle$$
$$G_r = \langle \cos\theta, \sin\theta, 0 \rangle$$
$$G_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$N = G_r \times G_\theta = \langle 0, 0, r \rangle$$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & 4z & x^2 \end{vmatrix}$$

$$= -4\vec{i} - (2x)\vec{j} + 1\vec{k}$$

$$= \langle -4, -2x, -1 \rangle$$

$$= \langle -4, -8\cos\theta, -1 \rangle$$

$$\text{curl}(F) \cdot n = -4(0) - (8\cos\theta)(0) + 1(r) = r$$

$$\oint_{\partial C} \vec{F} \cdot dr = \iint \text{curl } F \cdot n \, dS = \int_0^{2\pi} \int_0^2 r \, dr \, d\theta$$

$$= \boxed{4\pi}$$

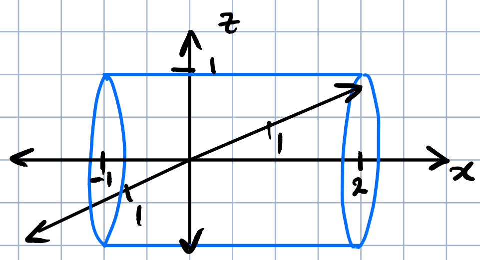
Exercise 7 DIVERGENCE THEOREM.

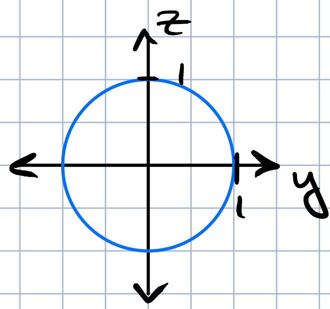
Let $\mathbf{F}(x,y,z) = \langle 3xy^2, xe^z, z^3+4x \rangle$ and $\mathscr{S}$ be a surface of the solid bounded by the cylinder $y^2+z^2=1$ and the planes $x=-1$ and $x=2$.

(i) Sketch the surface $\mathscr{S}$.

(ii) Use Divergence Theorem to compute $\displaystyle\int\int_{\mathscr{S}} \mathbf{F} \cdot d\mathbf{S}$

(i)





(ii) $\operatorname{div}(F) = 3y^2 + 0 + 3z^2$

$= 3y^2 + 3z^2$

Let's use cylindrical coords of the form:

$y = r\cos\theta$, $z = r\sin\theta$, $x = x$, $dV = r\,dr\,d\theta\,dx$

$\operatorname{div}(F) = 3r^2\cos^2\theta + 3r^2\sin^2\theta$

$= 3r^2$

$$\int_{-1}^{2}\int_{0}^{2\pi}\int_{0}^{1} 3r^2 \cdot r\,dr\,d\theta\,dx$$

$$= \int_{-1}^{2}\int_{0}^{2\pi} \frac{3}{4}r^4\Big|_0^1\,d\theta\,dx$$

$$= \left[\frac{3\pi}{2}x\right|_{-1}^{2} = \boxed{\frac{9\pi}{2}}$$