Final Exam

MATH 32b @ UCLA (SPRING 2021)

Assigned: June 07, 2021.

Instructions/Admonishment

1. SHOW ALL WORK.
   A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit.
2. Duration: 24 hours.
3. The following is my own work, without the aid of any other person.
   Signature: ____

Problem 01 Double Integral in Polar Coordinates.

Consider the intergal $\displaystyle\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}} dydx + \int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{|y|}^{\sqrt{1-y^2}} dxdy$

(i) Use the polar coordinates to combine the integrals into a single double integral.

(ii) Evaluate the integral.

Problem 02 Triple Integral in Spherical Coordinates.

Consider the solid $\mathscr{W}$ bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$.

(i) Find the spherical coordinate limits for the integral that calculates the volume of the region $\mathscr{W}$.

(ii) Evaluate the integral.

Problem 03 Change of Variables.

Let $\mathscr{R}$ be the trapezoid with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$ and $\mathscr{J} = \iint_{\mathscr{R}} \cos\left(\frac{y-x}{y+x}\right) dA$

(i) Give a change of variables u and v and find the image of the trapezoid $\mathscr{R}$ in the uv-plane using this transformation.

(ii) Compute $\mathscr{J}$.

Problem 04 Conservative Vector Fields.

The vector field $\mathbf{F}(x,y) = \left\langle \frac{-x}{(x^2+y^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2)^{\frac{3}{2}}} \right\rangle$ is defined on the region $\mathscr{D} = \{(x,y) \neq (0,0)\}$.

(i) Is $\mathscr{D}$ a simply connected region?

(ii) Show that $\mathbf{F}$ satisfies the cross-partials condition. Does this guarantee that $\mathbf{F}$ is conservative?

(iii) Show that $\mathbf{F}$ is conservative on $\mathscr{D}$ by finding a potential function.

Problem 05 Green's Theorem.

Consider the line integral $\mathscr{I} = \oint_{\mathscr{C}} xydy - y^2dx$ where $\mathscr{C}$ is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

(i) Find the line integral $\mathscr{I}$ using the circulation form of Green's Theorem.

(ii) Find the line integral $\mathscr{I}$ using the flux form of Green's Theorem.

Problem 06 Stokes' Theorem.

Let $\mathscr{C}$ be the curve given by the intersection of the plane $z = 2$ and the cone $z = \sqrt{x^2 + y^2}$ counterclockwise oriented as viewed from above. Let the vector field $\mathbf{F} = \langle x^2 - y, 4z, x^2 \rangle$. Use Stokes' theorem to find the circulation of $\mathbf{F}$ around $\mathscr{C}$ by integrating over

(i) the surface of the cone

(ii) the flat disk of radius 2 centered on the z-axis and lying in the plane $z = 2$.

Problem 07 Divergence Theorem.

Let $\mathbf{F}(x, y, z) = \left\langle 3xy^2, 3x^2y + 4x, xe^y \right\rangle$ and $\mathscr{S}$ be an oriented surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$ and $z = 2$.

(i) Explain why Divergence Theorem applies.

(ii) Use Divergence Theorem to compute $\displaystyle\int\int_{\mathscr{S}} \mathbf{F} \cdot d\mathbf{S}$.