32B Lect. 1 Midterm 1 January 31

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Section: 1E $= \begin{cases} 1a & \text{Tuesday with Eric Auld} \\ 1b & \text{Thursday with Eric Auld} \\ 1c & \text{Tuesday with Kyung Ha} \\ 1d & \text{Thursday with Kyung Ha} \\ 1e & \text{Tuesday with Khang Huynh} \\ 1f & \text{Thursday with Khang Huynh} \end{cases}$

Rules.

- There are **FOUR** problems; ten points per problem.
- There is an extra page at the end. You may also use the backs of pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone, pager,...

1	2	3	4	Σ
9	3	9	4	~~23~~ 25

$\iint \rho ($

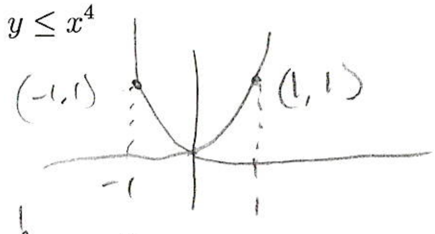
$X_{cm} = \dfrac{\iint x \, \rho(x,y) \, dA}{\text{mass } \iint \rho(x,y)}$ $\quad Y_{cm} = \dfrac{\iint y \, \rho(x,y) \, dA}{}$

(1) Find the center of mass for a homogeneous planar body occupying the region where

$$-1 \le x \le 1 \qquad \text{and} \qquad 0 \le y \le x^4$$

ρ is constant

$\rho = k \quad k = \text{constant}$



$$\text{Mass} = \int_{-1}^{1}\int_{0}^{x^4} k \, dy\,dx = \int_{-1}^{1} [ky]_0^{x^4} \, dx = \int_{-1}^{1} k x^4 \, dx = \left[\frac{kx^5}{5}\right]_{-1}^{1}$$

$$\text{mass} = k\left(\tfrac{1}{5}\right) - \left(k\left(-\tfrac{1}{5}\right)\right) = \frac{2k}{5}$$

$$M_x = \int_{-1}^{1}\int_{0}^{x^4} y k \, dy\,dx = \int_{-1}^{1} \left(\frac{ky^2}{2}\right)_{y=0}^{y=x^4} dx = \int_{-1}^{1} \frac{kx^8}{2} \, dx = \left[\frac{kx^9}{8}\right]_{-1}^{1}$$

2×9

$$= \frac{k(1)^9}{8} - \left(\frac{k(-1)^9}{8}\right) = \frac{2k}{8} = \frac{k}{4}$$

$$Y_{cm} = \frac{M_x}{m} = \frac{\left(\frac{k}{4}\right)}{\left(\frac{2k}{5}\right)} = \frac{5k}{8k} = \frac{5}{8}$$

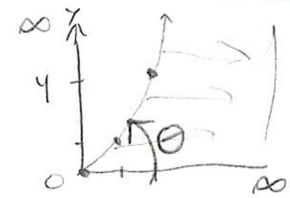
$$M_y = \int_{-1}^{1}\int_{0}^{x^4} x k \, dy\,dx = \int_{-1}^{1} [xky]_0^{x^4} \, dx = \int_{-1}^{1} kx^5 \, dx = \left[\frac{kx^6}{6}\right]_{-1}^{1}$$

$$= \frac{k(1)}{6} - \left(\frac{k(1)}{6}\right) = 0$$

$$X_{cm} = \frac{0}{\left(\frac{2k}{5}\right)} = 0$$

C.O.M is $\left(0, \frac{5}{8}\right)$

$\frac{d}{dx} \sec x = \sec x \tan x$



(2) Evaluate the integral

$$\int_0^\infty \int_{\sqrt{y}}^\infty \frac{y}{(x^2+y^2)^2}\, dx\, dy \qquad r\,dr\,d\theta$$

by converting to polar coordinates.

$\sqrt{y} \le x \le \infty \quad 0 \le y \le \infty$

$\frac{r\sin\theta}{(r^2)^2} = \frac{\sin\theta}{r^3}$

$\iint \frac{\sin\theta}{r^3}\, r\,dr\,d\theta = \iint \frac{\sin\theta}{r^2}\, dr\,d\theta$

$\int_0^{2\pi} \int_0^{\frac{\sin\theta}{\cos^2\theta}} \frac{\sin\theta}{r^2}\, dr\,d\theta$

$r\sin\theta \le r^2\cos^2\theta$

$r \ge \tan\theta \sec\theta$

$0 \le r\sin\theta \le \infty$

$r \ge 0$

$\int_0^\infty \int_{\sqrt{y}}^\infty \frac{y}{(x^2+y^2)^2}\, dx\,dy$

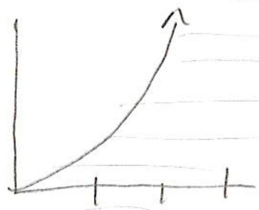
$= \int_0^{\pi/2} \int_{\frac{\sin\theta}{\cos\theta}}^\infty \frac{\sin\theta}{r^2}\, dr\,d\theta$

$= \int_0^{\pi/2} \left[-\sin\theta\, r^{-1}\right]_{\frac{\sin\theta}{\cos\theta}}^\infty d\theta$

$= \int_0^{\pi/2} \cos^2\theta\, d\theta$

$= \int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta$

$=$



as $x \to \infty$, $0 \le \theta \le \frac{\pi}{2}$

$y = x^2$

$x\sin\theta = r^2\cos^2\theta$

$r = \frac{\sin\theta}{\cos^2\theta}$

$0 < x$

$0 < y$

$y \le x^2$

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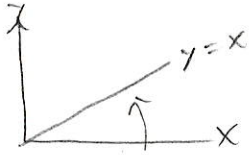
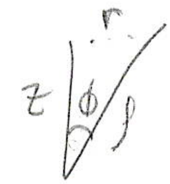
$\phi \quad 0 \le \rho\sin\phi\cos\theta \le \rho\sin\phi\sin\theta_y$



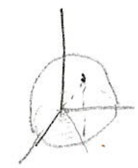


(3) Determine the volume of the region defined by the following inequalities

$$0 \le x \le y \qquad \text{and} \qquad \overset{\text{sphere}}{x^2 + y^2 + z^2 \le 1}$$

$$\int\int\int 1 \, dV \qquad \rho^2 \le 1$$

$$\rho \le 1$$

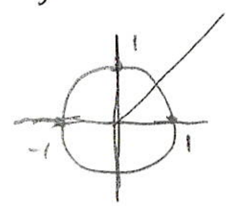


$$\cancel{\int_{\theta=0}^{\pi/4}} \int_{\phi=0}^{\pi} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$r \, dr \, dz \, d\theta$

$*$ should be $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$



$$= \int_0^{\pi/4} \int_{\phi=0}^{\pi} \left[\frac{\rho^3}{3} \sin\phi\right]_0^1 d\phi \, d\theta$$

$$= \int_0^{\pi/4} \int_0^{\pi} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{\pi/4} \left[\frac{-\cos\phi}{3}\right]_0^{\pi} d\theta = \int_0^{\frac{\pi}{4}} \left[\underbrace{-\frac{\cos(\pi)}{3}}_{\frac{1}{3}} - \underbrace{\left(-\frac{\cos(0)}{3}\right)}_{-\frac{1}{3}}\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{2}{3} \, d\theta = \left[\frac{2}{3}\theta\right]_0^{\frac{\pi}{4}} = \frac{2\pi}{12} - 0 = \frac{\pi}{6}$$

To spherical

$0 \le \rho\cos\phi \cos\theta \le \rho\cos\phi\sin\theta$

$0 \le \cos\theta \le \sin\theta$

$0 \le \cot\theta \le 1$

$\frac{\pi}{2} \ge \theta \ge \frac{\pi}{4}$

$\rho^2 \le 1$

$0 \le \rho \le 1$

$0 \le x$

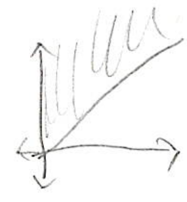
$$V = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 \int_0^{\pi} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$

when is sine $\ge$ cosine?

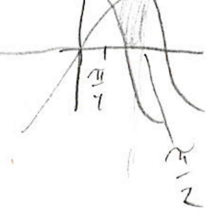
$\boxed{y \ge x}$

$\tan\theta \ge 1$

$r\sin\theta \ge r\cos\theta$



$x \le y$



(4) Consider the region $\mathcal{R}$ defined by the inequalities

$$0 \le x^2 \le y \le z \le 3$$

(a) Determine the volume of the region $\mathcal{R}$.
(b) Determine the area of the cross-section of $\mathcal{R}$ lying in the plane $y = 1$.

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(a) $\int\int\int [0 \le x^2 \le y \le z \le 3]\, dz\, dy\, dx$

Algebraic method

$\int\int [0 \le x^2 \le y] \int [y \le z \le 3]\, dz\, dy\, dx$

$\int\int [0 \le x^2 \le y][y \le 3] \int_y^3 dz\, dy\, dx$

$\int [x^2 \ge 0] \int [y \ge x^2][y \le 3] \int_y^3 dz\, dy\, dx$

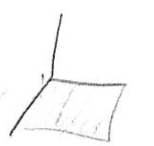
$\int [x^2 \ge 0][x^2 \le 3] \int_{x^2}^3 \int_y^3 dz\, dy\, dx$

$= \int_0^{\sqrt{3}} \int_{x^2}^3 \int_y^3 dz\, dy\, dx = \int_0^{\sqrt{3}} \int_{x^2}^3 [z]_y^3\, dy\, dx$

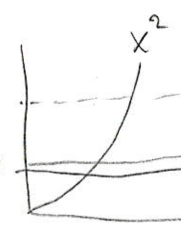
$= \int_0^{\sqrt{3}} \int_{x^2}^3 (3 - y)\, dy\, dx = \int_0^{\sqrt{3}} \left[3y - \frac{y^2}{2}\right]_{x^2}^3 dx = \int_0^{\sqrt{3}} \left[9 - \frac{9}{2} - \left(3x^2 - \frac{x^4}{2}\right)\right] dx$

$= \int_0^{\sqrt{3}} \frac{9}{2} - 3x^2 + \frac{x^4}{2}\, dx = \left[\frac{9}{2}x - x^3 + \frac{x^5}{10}\right]_0^{\sqrt{3}} = \frac{9}{2}\sqrt{3} - 3\sqrt{3} + \frac{(\sqrt{3})^5}{10}$

(b) Area $= \iint_R 1\, dA$



$\int \left(\underbrace{\int \quad dx}_{\text{cross sectional area}} \right) dy$ × thickness



$\int_0^1 \int_0^{\sqrt{y}}$

$\int_0^1 \int_0^{\sqrt{3}} 1\, dx$

$0 \le x^2 \le y \le 3$

$\int_0^1 \int_0^{\sqrt{3}} 1\, dx\, dy$

$\int \int [y \le z][z \le 3] \int [0 \le x^2 \le y]$

$[\quad [0 \le \sqrt{y}] \int_0^{\sqrt{y}}$

when $x^2 = 3$, $x = \sqrt{3}$

$= \int_0^1 [x]_0^{\sqrt{3}}\, dy$

$= \int_0^1 \sqrt{3}\, dy = [\sqrt{3}\, y]_0^1 = \sqrt{3}$

$\int_0^1 \int_0^{\sqrt{y}} 1\, dx\, dy$

$= \int_0^1 [\sqrt{y}]\, dy$

$= \left[\frac{2}{3} y^{3/2}\right]_0^1 = \frac{2}{3}(1) = \frac{2}{3}$