Math 32b – Lecture 3 – Winter 2022
Final Exam
Due via Gradescope no later than 8am Pacific daylight time on Thursday Mar/17/2022.

Sign and submit the following honor statement:

> *I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.*

Signed:

Print name:

This exam contains 10 pages (including this cover page) and 9 problems. There are a total of 135 points available.

- Use extra pages as you need them.
- Attempt all questions.
- The problems are in no particular order.
- The work submitted must be entirely your own: you may not discuss with anyone else, except that...
- You may email the instructor `killip@math.ucla.edu` with any queries about what the questions are asking.
- This exam is open book. You may use your notes, the textbook, and any online resource that does not involve interaction with another person.
- **Posting problems to online forums or "tutoring" websites counts as interaction with another person so is strictly forbidden.**

1. (15 points) • Which of the following is a correct expression of the area of a certain triangle?

$$\mathbf{A}: \int_0^x \int_0^1 1\,dy\,dx \qquad \mathbf{B}: \int_0^1 \int_0^x 1\,dy\,dx$$

$$\mathbf{C}: \int_0^1 \int_0^x 1\,dx\,dy \qquad \mathbf{D}: \int_0^x \int_0^1 1\,dx\,dy$$

• Let $\mathcal{R}$ denote the rectangle with vertices

$$(0,0,0), \quad (0,1,0), \quad (3,0,4), \quad \text{and} \quad (3,1,4).$$

What is the value of $\iint_{\mathcal{R}} 1\,dS$?

$\mathbf{A}: \sqrt{3}$ $\quad$ $\mathbf{B}: 2$ $\quad$ $\mathbf{C}: \sqrt{5}$ $\quad$ $\mathbf{D}: 3$ $\quad$ $\mathbf{E}: 4$ $\quad$ $\mathbf{F}: 5$

• The Jacobian for the change of variables $x(u,v) = u - 5v$, $y(u,v) = 2v$ is

$\mathbf{A}: 2$ $\quad$ $\mathbf{B}: 5$ $\quad$ $\mathbf{C}: -1$ $\quad$ $\mathbf{D}: 10$ $\quad$ $\mathbf{E}$: undefined

2. (15 points) A circular pond is described by the inequalities

$$x^2 + y^2 \leq 1 \quad \text{and} \quad xy - 1 \leq z \leq 0$$

Determine the volume of the pond.

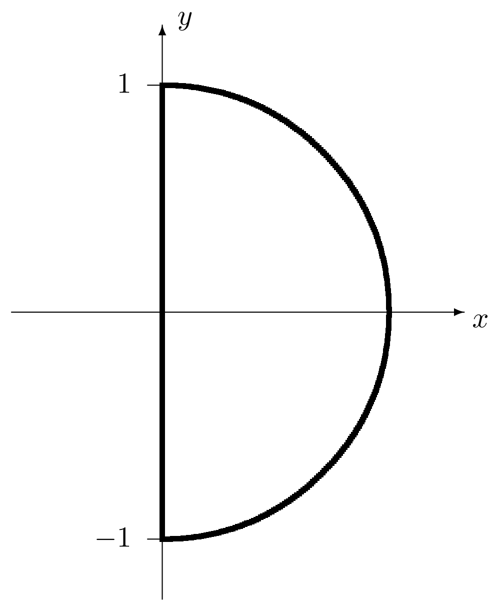
3. (15 points) I traverse a bow-tie shaped path by walking in straight lines between the points

$$(-1,-1), \quad (1,1), \quad (1,-1), \quad (-1,1), \quad \text{and then back to} \quad (-1,-1).$$

Demonstrate the use of Green's Theorem by using it to compute the work I do against the field

$$\vec{F} = \begin{bmatrix} x\cos(y)\sin(y) \\ x^2\cos^2(y) \end{bmatrix}$$

4. (15 points) A fine wire of density 100 g/m is formed into the following D shape comprised of a line segment and a semicircle:



Labels are in meters. Determine the moment of inertia for rotations about the y axis

5. (15 points) An Egyptian-style pyramid $\mathcal{P}$ has corners at the points

$$(-3,-3,0), \quad (-3,+3,0), \quad (+3,+3,0), \quad (+3,-3,0), \quad \text{and} \quad (0,0,7).$$

We orient the faces of the pyramid outwards.

(a) Find limits of integration so as to write $\iiint_{\mathcal{P}} f(x,y,z)\,dV$ as an iterated integral.

(b) We wish to find the flux of the vector field

$$\vec{F} = \begin{bmatrix} 1+z \\ 2+z^2 \\ 3+z^3 \end{bmatrix}$$

through the four triangular faces of the pyramid. Use the divergence theorem to relate this to an integral over the (square) bottom of the pyramid.

(c) Use your result from (b) to compute the sought-after flux.

6. (15 points) For this problem, let $A(\phi) = \dfrac{1}{\sqrt{\sin^2(\phi) - \cos^2(\phi)}}$.

Convert the following integral, given in spherical polar coordinates, to cylindrical coordinates.

$$\int_{\pi/4}^{3\pi/4} \int_0^{2\pi} \int_0^{A(\phi)} \sin(\phi)\, d\rho\, d\theta\, d\phi$$

You are **not** expected to evaluate it!

7. (15 points) Consider a pipe parallel with the z-axis given by $x^2 + y^2 \leq 2$. The velocity of a fluid flowing in the pipe is given by

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 4 - x^2 - y^2 \end{bmatrix}$$

(a) Determine $\oint (\nabla \times \vec{v}) \cdot d\vec{r}$ around the loop $x = \cos(\theta)$, $y = \sin(\theta)$, $z = 0$, $0 \leq \theta \leq 2\pi$.

(b) Demonstrate the use of Stokes' Theorem to confirm your answer to part (a).
Be explicit about the surface you use and its orientation!

8. (15 points) Use the change of variables

$$x = e^u \qquad y = ve^u$$

to evaluate the following integral:

$$\int_7^9 \int_x^{x\ln(x)} \frac{y}{x^3}\, dy\, dx$$

9. (15 points) Let $\mathcal{R}$ denote the three-dimensional region where $x^2 + y^2 > 1$ and consider the scalar function

$$f(x, y, z) = \ln(x^2 + y^2) \qquad \text{defined on } \mathcal{R}.$$

(a) Is $\mathcal{R}$ simply connected?

(b) Determine $\vec{F} = \nabla f$

(c) Determine $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$

(d) Compute $\oint \vec{F} \cdot d\vec{r}$ around the circle defined by

$$(x - 3)^2 + y^2 = 1 \quad \text{and} \quad z = 0$$

oriented clockwise when viewed from above (i.e. from a point where $z > 0$).

(e) Is $\vec{F}$ conservative? Justify your answer.