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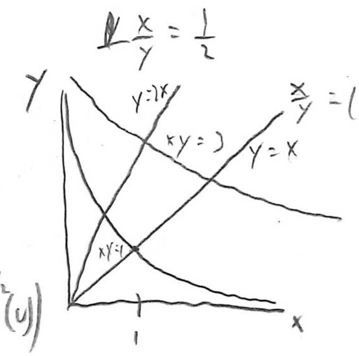
Instructions: You have 50 minutes to complete this exam. There are five problems worth a total of 50 points. This exam is closed book and closed notes, and calculators are not allowed. You must justify your answers and show all of your work to receive full credit. Simplify your answers as much as possible. You may lose points for answers that are not simplified. Write your solutions in the space below each question. If your answer continues onto another page, write an easily visible note under the original question. You may use the last two pages of the exam for scratch work.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Let Ω be the region in the first quadrant of $\mathbb{R}^2$ bounded by $y = x$, $y = 2x$, $xy = 1$, and $xy = 3$. Evaluate

$$\int_\Omega \frac{x}{y}\, dA.$$



$u = xy \qquad v = \frac{x}{y}$

$uv = x^2$

$x = \sqrt{uv}$

$y = \frac{u}{x} = \frac{u}{\sqrt{uv}}$

$= u(uv)^{-\frac{1}{2}}$

$G(u,v) = \left(\sqrt{uv}, \frac{u}{\sqrt{uv}}\right)$

$\mathrm{Jac}(G) = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \left(\frac{1}{2}(uv)^{-\frac{1}{2}}(v)\right)\left(-\frac{1}{2}u^2(uv)^{-3/2}\right)$

$- \left(\frac{1}{2}(uv)^{-\frac{1}{2}}(u)\right)\left((uv)^{-\frac{1}{2}} - \frac{1}{2}u(uv)^{-3/2}(v)\right)$

$= -\frac{u^2 v}{4}(uv)^{-2} - \frac{u}{2}(uv)^{-1} + \frac{u^2 v}{4}(uv)^{-2}$

$= \frac{-u}{2uv} = -\frac{1}{2v}$

$\int_\Omega \frac{x}{y}\, dA = \int_{\frac{1}{2}}^{1}\int_{1}^{3} v\left(\frac{1}{2v}\right) du\, dv = \frac{1}{2}(2)\left(\frac{1}{2}\right) = \boxed{\frac{1}{2}}$

2. Let $f(x, y, z) = x^2$ and $\mathbf{F} = (x - y, y - z, z)$.

(a) (5 points) Compute

$$\int_C f(x, y, z)\, ds,$$

where C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$ in $\mathbb{R}^3$.

$\vec{r}(t) = (t, 2t, 3t) \quad 0 \le t \le 1 \qquad \vec{r}'(t) = (1, 2, 3) \qquad \|\vec{r}'(t)\| = \sqrt{1+4+9} = \sqrt{14}$

$\int_C f(x, y, z)\, ds = \int_0^1 f(\vec{r}(t))\, \|\vec{r}'(t)\|\, dt = \sqrt{14} \int_0^1 t^2\, dt = \sqrt{14} \left| \frac{t^3}{3} \right|_0^1 = \boxed{\frac{\sqrt{14}}{3}}$

(b) (5 points) Compute

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the line segment from $(0, 0, 0)$ to $(1, 2, 2)$ in $\mathbb{R}^3$.

$\vec{r}(t) = (t, 2t, 2t) \quad 0 \le t \le 1 \qquad \vec{r}'(t) = (1, 2, 2)$

$\int_C F \cdot dr = \int_0^1 F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_0^1 (-t, 0, 2t) \cdot (1, 2, 2)\, dt = \int_0^1 (-t + 4t)\, dt = \int_0^1 3t\, dt$

$= \left| \frac{3t^2}{2} \right|_0^1 = \boxed{\frac{3}{2}}$

3. (10 points) Let $\mathcal{C}$ be the curve in $\mathbb{R}^2$ parametrized by

$$\mathbf{r}(t) = \left(e^{t^7}, t^6 + 4t^3 - 1, t^4 + (t - t^2)e^{\sin t}\right),$$

where $0 \le t \le 1$, and let

$$\mathbf{F} = \left(\frac{z}{x} + y, x + z, \ln x + y + 2z\right).$$

Evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

$$\operatorname{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{z}{x}+y & x+z & \ln x + y + 2z \end{vmatrix} = \left(1-1, -\frac{1}{x} + \frac{1}{x}, 1-1\right) = \vec{0}, \text{ so F is conservative}$$

$$F = \nabla f$$

$$\left(\frac{z}{x}+y, x+z, \ln x + y + 2z\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\frac{\partial f}{\partial x} = \frac{z}{x} + y$$

$$\frac{\partial f}{\partial y} = x + z$$

$$\frac{\partial f}{\partial z} = \ln x + y + 2z$$

$$\partial f = \left(\frac{z}{x} + y\right) dx$$

$$f(x,y,z) = z\ln x + xy + g(y,z)$$

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = \cancel{x} + z$$

$$\partial g = z\, dy$$

$$g(y,z) = zy + h(z)$$

$$f(x,y,z) = z\ln x + xy + zy + h(z)$$

$$\frac{\partial f}{\partial z} = \ln x + y + \frac{\partial h}{\partial z} = \cancel{\ln x} + \cancel{y} + 2z$$

$$\partial h = 2z\, \partial z$$

$$h = z^2 + C$$

$$f(x,y,z) = z\ln x + xy + zy + z^2 + C$$

$$\int_C F \cdot dr = f(r(1)) - f(r(0)) = f(e, 4, 1) - f(1, -1, 0) = (\ln e + 4e + 4 + 1) - (-1) = \boxed{7 + 4e}$$

4. (10 points) Compute the surface area of the portion of the paraboloid

$$z = 2x^2 + 2y^2$$

where $0 \le z \le 4$.

$x^2+y^2 = \frac{2}{2}$ $\qquad r^2 = \frac{2}{2}$ $\qquad z = 2r^2$

$G(r,\theta) = (r\cos\theta, r\sin\theta, 2r^2)$ $\quad 0 \le r \le \sqrt{2}$ $\quad 0 \le \theta \le 2\pi$

$T_r = (\cos\theta, \sin\theta, 4r)$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

$N(r,\theta) = T_r \times T_\theta = (-4r^2\cos\theta, -4r^2\sin\theta, r\cos^2\theta + r\sin^2\theta) = (-4r^2\cos\theta, -4r^2\sin\theta, r)$

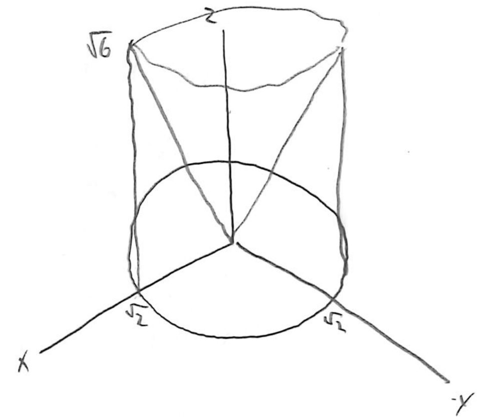
$\|N\| = \sqrt{16r^4\cos^2\theta + 16r^4\sin^2\theta + r^2} = \sqrt{16r^4 + r^2} = r\sqrt{16r^2+1}$

Surface area $= \frac{1}{32}\int_0^{2\pi}\int_0^{\sqrt{2}} 32r\sqrt{16r^2+1}\,dr\,d\theta = \frac{2\pi}{32}\left.\frac{2(16r^2+1)^{3/2}}{3}\right|_0^{\sqrt{2}} = \boxed{\frac{\pi}{24}\left(33^{3/2}-1\right)}$

5. (10 points) Let $\mathcal{S}$ denote the portion of the cone $z^2 = 3x^2 + 3y^2$ lying over the disk $x^2 + y^2 \le 2$. Calculate

$$\iint_{\mathcal{S}} x^2 \, dS.$$

$G(r,\theta) = (r\cos\theta, r\sin\theta, r\sqrt{3}) \quad 0 \le r \le \sqrt{2} \quad 0 \le \theta \le 2\pi$



$T_r = (\cos\theta, \sin\theta, \sqrt{3})$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

$N(r,\theta) = T_r \times T_\theta = (-r\sqrt{3}\cos\theta, r\sqrt{3}\sin\theta, r\cos^2\theta + r\sin^2\theta)$

$= (-r\sqrt{3}\cos\theta, -r\sqrt{3}\sin\theta, r)$

$\|N\| = \sqrt{3r^2\cos^2\theta + 3r^2\sin^2\theta + r^2} = \sqrt{4r^2} = 2r$

$\iint_S x^2 \, dS = \int_0^{2\pi}\int_0^{\sqrt{2}} r^2\cos^2\theta (2r)\, dr\, d\theta = 2\int_0^{2\pi}\int_0^{\sqrt{2}} r^3\cos^2\theta\, dr\, d\theta = 2\int_0^{2\pi} \left. \frac{r^4}{4}\cos^2\theta \right|_0^{\sqrt{2}} d\theta$

$= \frac{1}{2}\int_0^{2\pi} 4\cos^2\theta\, d\theta = 2\int_0^{2\pi}\cos^2\theta\, d\theta = \boxed{2\pi}$

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