MATH 32B
SECOND PRACTICE MIDTERM EXAMINATION

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination.*

Name:________________________ Section:__________________

#1	#2	#3	#4	#5		Total

Problem 1. Let D be a sector on the xy-plane with the central angle $2\theta_0$ and radius R, symmetric with respect to the y-axis. Suppose the mass density is $\delta(x, y) = x^2$. Find coordinates of the center of mass.

SOLUTION: Since the region is symmetric with respect to the y-axis and $\delta(-x, y) = \delta(x, y)$, it follows that the center of mass lies on the y-axis (i.e., the x-coordinate of the center of mass is 0).

The mass m and the moment M_x with respect to x-axis are given by

$$
\begin{aligned}
m &= \int_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}+\theta_0} \int_0^R r^2 \cos^2\theta \cdot r dr d\theta = \\
&= \frac{R^4}{4} \cdot \int_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}+\theta_0} \frac{1 + \cos 2\theta}{2} d\theta = \\
&= \frac{R^4}{4} \cdot \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right] \Big|_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}+\theta_0} = \\
&= \frac{R^4}{4} \cdot \left[\theta_0 + \frac{-\sin 2\theta_0 - \sin 2\theta_0}{4}\right] = \\
&= \frac{R^4}{4} \cdot \left[\theta_0 - \frac{\sin 2\theta_0}{2}\right].
\end{aligned}
$$

$$
\begin{aligned}
M_x &= \int_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}+\theta_0} \int_0^R r^2 \cos^2\theta \cdot r \sin\theta r dr d\theta = \\
&= \frac{R^5}{5} \cdot \int_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}+\theta_0} \cos^2\theta \cdot \sin\theta d\theta = \\
&= -\frac{R^5}{5} \cdot \frac{1}{3} \cos^3\theta \Big|_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}+\theta_0} = \frac{R^5}{15} \left(\cos^3\left(\frac{\pi}{2} - \theta_0\right) - \cos^3\left(\frac{\pi}{2} + \theta_0\right)\right) = \\
&= \frac{2R^5}{15} \sin^3\theta_0.
\end{aligned}
$$

Thus, the y-coordinate of the center of mass is

$$
\bar{y} = \frac{M_x}{m} = \frac{8}{15} R \cdot \frac{\sin^3\theta_0}{\theta_0 - \frac{\sin 2\theta_0}{2}} = \frac{8}{15} R \cdot \frac{\sin^3\theta_0}{\theta_0 - \sin\theta_0 \cdot \cos\theta_0}.
$$

Problem 2. Compute the integral

$$\iint_D e^{16x^2+25y^2}\,dxdy,$$

where D is the interior of the ellipse given by

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{4}\right)^2 \le 1.$$

SOLUTION: Use the change of variables $u = \frac{x}{5}$, $v = \frac{y}{4}$, the region of integration is described by $u^2 + v^2 \le 1$ and the Jacobian is $\frac{\partial(x,y)}{\partial(u,v)} = 20$. Then

$$I = \iint e^{400\cdot(u^2+v^2)} \cdot 20dudv$$

Further changing to polar coordinates (r, θ) we get

$$I = 20\int_0^{2\pi}\int_0^1 e^{400r^2} rdrd\theta = 40\pi \cdot \frac{1}{2}\int_0^1 e^{400t}dt = \frac{20\pi}{400}(e^{400} - 1) = \frac{\pi(e^{400} - 1)}{20}.$$

Here $t = r^2$.

Problem 3. Prove that

$$\text{curl}(f \cdot \mathbf{F}) = f \cdot \text{curl}(\mathbf{F}) + (\nabla f) \times \mathbf{F}$$

for any function f and any vector field $\mathbf{F}$.

SOLUTION: This can be checked using a direct computation. Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$\text{curl}(f\mathbf{F}) = \langle (fR)_y - (fQ)_z,\ (fP)_z - (fR)_x,\ (fP)_y - (fQ)_x \rangle.$$

Simplify the expression for the first component using the product rule:

$$(fR)_y - (fQ)_z = f(R_y - Q_z) + f_y R - f_z Q.$$

Here $f(R_y - Q_z)$ is the first component of $f \cdot \text{curl}(\mathbf{F})$ and $f_y R - f_z Q$ is the first component of $(\nabla f) \times \mathbf{F}$.

The equality of the other two components can be checked similarly.

Problem 4. Let $\mathbf{F} = (\sin x^2 + y, x + z, y)$ be a vector field and C be the curve

$$\begin{aligned} x &= \sin t \\ y &= \cos t \\ z &= t^2 \end{aligned}$$

for $-\pi \leq t \leq \pi$.

(a) Is the vector field $\mathbf{F}$ conservative?

SOLUTION: The vector field is defined for all values of x, y, z. Compute curl($\mathbf{F}$) :

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ \sin x^2 + y & x + z & y \end{vmatrix} = \mathbf{0}.$$

Since curl($\mathbf{F}$) $= 0$ on a simply-connected region (the whole plane), it follows that the vector field is conservative.

(b) Compute the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

SOLUTION: Notice that the curve $\mathbf{r}(t) = \langle \sin t, \cos t, t^2 \rangle$ has the property that

$$\mathbf{r}(-\pi) = \langle 0, -1, 1 \rangle = \mathbf{r}(\pi).$$

Thus, the curve we are given is a closed curve. Since the vector field is conservative, it has path-independence property. Thus, the value of the integral around a closed curve is equal to 0.

Problem 5. Multiple-choice questions:

(1) Let $\vec{r} = \langle x, y, z\rangle$ and $r = |\vec{r}|$. Which of the following is *not* correct:
  (a) $\nabla \cdot (r \cdot \vec{r}) = 4r$
  (b) $\nabla \cdot (\vec{r}) = 3$
  (c) **FALSE** : $\nabla \left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^2}$
  (d) $\nabla(r^2) = 2 \cdot (x + y + z)$

  SOLUTION: Statement (c) is not correct:

$$\nabla \left(\frac{1}{r}\right) = \nabla(r^{-1}) \overset{\text{Chain Rule}}{=} -r^2 \cdot \nabla r = -r^2 \cdot \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\rangle = \frac{\vec{r}}{r^3}.$$

  The other statements are correct and can be verified by a direct computation.

(2) Let $\overrightarrow{F} = \langle F_1(x, y), F_2(x, y)\rangle$ be a vector field defined everywhere on the plane and such that $F_1 = f_x$ and $F_2 = f_y$. Circle the correct statement from the statements below:
  (a) The flux of this vector field across any circle centered at $(0, 0)$ is 0.
  (b) **TRUE** The circulation of this vector field along any circle centered at $(0, 0)$ is 0.
  (c) The divergence of this vector field is 0.
  (d) $(F_1)_x = (F_2)_y$ for all points on the plane.

  SOLUTION: We are given that $\overrightarrow{F} = \langle f_x, f_y\rangle$ everywhere on the plane. This means that $\overrightarrow{F}$ is conservative. Therefore, the integral of this vector field along any closed curve (and in particular, along a circle centered at $(0, 0)$) is 0. Thus, statement (b) is correct. The other statements are false.

(3) Circle the correct statement:
  (a) **TRUE** If $\overrightarrow{r}(t) = \langle x(t), y(t)\rangle$, $t \in [0, 1]$ is a parametrization of a curve C, then $\overrightarrow{r}(t) = \langle x(1 - t), y(1 - t)\rangle$ is a parametrization of the same curve traversed in the opposite direction.
  (b) If $\overrightarrow{r}(t)$ is a parametrization such that $|\overrightarrow{r}(t)| = 1$ for all t, then the curve is a segment of a straight line.
  (c) The value of the line integral

$$\int_C f ds$$

  depends on the chosen parametrization of the curve C.
  (d) Let $\overrightarrow{F}(x, y)$ be a vector field on the plane which is vertical at all points. If $\overrightarrow{F}$ is conservative with potential function f, then vertical lines are level sets of f.

  SOLUTION: Statement (a) is correct.

(4) Let G be a linear transformation from the (u, v) plane to the (x, y)-plane. Suppose that the u-axis is mapped to the line $y = x$ and the v-axis is moved to the y-axis by this transformation. Which of the following is true:

(a) **TRUE** The transformation moves all vertical lines to vertical lines;

(b) For any integrable function $f(x, y)$, the following is true:

$$\iint_{G(D)} f(x,y)dxdy = \iint_D f(x(u,v), y(u,v))dudv,$$

where $G(D)$ is the image of a region D on the uv-plane.

(c) The Jacobian of the transformation G is $J(x, y)$ such that $J(0, 0) = 2$.

(d) The line $u = v$ is mapped to the x-axis by this transformation.

Solution: Since this is a linear transformation, we have

$$\begin{aligned} x &= Au + Bv \\ y &= Cu + Dv \end{aligned}$$

for some A, B, C, D.

Since the u-axis (i.e., points with $v = 0$) is moved to the line $x = y$ we have

$$x = Au = y = Cu,$$

which implies that $A = C \neq 0$.

Since the v-axis (i.e, points with $u = 0$) is moved to y-axis (i.e., points with $x = 0$), we have

$$x = Bv = 0,$$

which implies that $B = 0$. Thus, the linear transformation has the form

$$\begin{aligned} x &= Au \\ y &= Au + Dv \end{aligned}$$

where $A \neq 0$. The Jacobian of this linear transformation is AD. Since G is invertible $AD \neq 0$, i.e., both $A, D \neq 0$. Answers (b) and (c) are both wrong.

To check (a), write equation for a vertical line: $u = c$. This is mapped to $x = Ac$, $y = Ac + Dv$. Since A and c are constants, x is fixed. If $D \neq 0$, y takes all possible values, and $u = c$ is mapped to the vertical line $x = AC$. Thus, (a) is correct.

To check (d), substitute $v = u$ into the formulas for the transformation. We get $x = Au$, $y = (A + D)u$. This is not the x-axis (unless $A = -D$). Thus, statement (d) is wrong.

(5) Circle the correct statement:

(a) Let $\overrightarrow{F} = f(r,\theta)\cdot\overrightarrow{r}$ be a vector field on the plane which is radial. Then $\text{curl}(\overrightarrow{F}) = 0$.

(b) **TRUE** If a vector field defined everywhere on the plane has path-independence property, then the vector field is conservative.

(c) If $\text{curl}(\overrightarrow{F}) = 0$ on the domain of definition of $\overrightarrow{F}$, then $\overrightarrow{F}$ is conservative.

(d) Suppose that $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 1$, where C is a unit circle on the plane. Then for any path C' surrounding the origin of coordinate system we have $\int_{C'} \overrightarrow{F} \cdot d\overrightarrow{r} = 1$.

Solution: Statement (b) is TRUE since the whole plane is simply-connected, and for simply-connected regions, the path-independence property implies that the vector field is conservative.
The other statements are wrong (one can construct a counterexample for every statement).