21W-MATH32B-4 MIDTERM 2

MATTHEW GUAN

TOTAL POINTS

30 / 30

QUESTION 1

1 Honor statement 0 / 0

✓ + **0 pts** **Complete**

QUESTION 2

2 Question 1 6 / 6

✓ + **6 pts** **Correct**

- **2 pts** No explanation (literally only equations and symbols without any context)

+ **2 pts** Correct integral for length($$\mathscr{C}$$)

+ **2 pts** Correct integral for $$\int_{\mathscr{C}} \mathbf{F} \cdot d \mathbf{r}$$

+ **2 pts** Correct final answer

+ **1 pts** Small mistake in integral for length($$\mathscr{C}$$)

+ **1 pts** Small mistake in integral for Setting up correct integral for $$\int_{\mathscr{C}} \mathbf{F} \cdot d \mathbf{r}$$

QUESTION 3

3 Question 2 7 / 7

✓ - **0 pts** **Correct $$\frac{125\pi}{6} $$**

Bounds for the region in spherical coordinates (3 marks total)

- **1 pts** $$\rho$$ bounds incorrect

- **1 pts** $$\phi$$ bounds incorrect

- **1 pts** $$\theta$$ bounds incorrect

Setting up and solving the integral (4 marks total)

- **2 pts** Did not use $$dV=\rho^2\sin(\phi)d\rho d\phi d\theta$$.

- **1 pts** Did not correctly convert the integrand into spherical coordinates.

- **1 pts** Error in calculating the final integral.

QUESTION 4

4 Question 3 8 / 8

✓ + **8 pts** **Everything correct**

+ **2 pts** Correct bounds for r: 1 to 2

+ **2 pts** Correct bounds for theta: 0 to 2pi

+ **2 pts** Correct jacobian r+r*cos(theta)

+ **1 pts** Correct integrand 2r

+ **1 pts** Correct final answer 28pi/3

QUESTION 5

5 Question 4 9 / 9

✓ + **1 pts** **Attempts to parameterize $$S$$**

✓ + **2 pts** **Correct choice of $$G(x,\theta) = (x,\cos\theta,\sin\theta)$$ _(or equivalent)_**

✓ + **1 pts** **Correct limits for $$\theta$$:**
*** Either $$0\leq \theta\leq \pi$$ for top half**
*** Or $$\pi\leq \theta\leq 2\pi$$ for bottom half**
(or equivalent)

✓ + **1 pts** **Correct limits for $$x$$: $$\sin\theta-1\leq x\leq 1-\sin\theta$$ _(or equivalent)_**

✓ + **2 pts** **Correctly computes (or cites) normal vector:**

*** Either $$\mathbf N = \langle0,-\cos\theta,-\sin\theta\rangle$$ for top half**
*** Or $$\mathbf N = \langle0,\cos\theta,\sin\theta\rangle$$ for bottom half**
(or equivalent)

+ **1 pts** _(Partial credit)_ Error in normal computation or correctly computes normal, but orientation incorrect.

✓ + **2 pts** **Correctly computes the flux to be:**
*** Either $$4-2\pi$$ for the top half**
*** Or $$4+2\pi$$ for the bottom half**

(Credit awarded if errors correctly carried)

+ **1 pts** _(Partial credit)_ Minor error in computation or added superfluous flux integrals to answer.

+ **0 pts** No credit due.

+ **1 pts** _(Partial credit)_ Computes normal of incorrect parameterization.

Math 32B - Lecture 4
Winter 2021
Midterm 2
Due 2/25/2021 before 10am

Sign and submit the following honor statement:

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signed:	Math G
Print name:	Matthew Guan

This exam contains 5 pages (including this cover page) and 4 problems. There are a total of 30 points available.

- Attempt all questions.
- Solutions must be uploaded to Gradescope before 10am Pacific Time on February 25^{th}.
  - Include extra pages as you need them.
  - You may complete the problems on a printout of this exam, blank paper, or a tablet/iPad.
  - If you handwrite your solutions, please make sure your scan is clearly legible.
  - Solutions should be written clearly, in full English sentences, defining all variables, showing all working, and giving units where appropriate.
- The work submitted must be entirely your own: you may not collaborate or work with anyone else to complete the exam.
- This exam is open book. You may use your notes, the textbook, and any online resource that does not involve interaction with another person.
- **Posting problems to online forums or "tutoring" websites counts as interaction with another person so is strictly forbidden.**

1 Honor statement **0 / 0**

✓ **+ 0 pts** **Complete**

1. **(6 points)** Let $\mathscr{C}$ be the curve with parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ for $-4\pi \leq t \leq 4\pi$. Find the value of the constant C that gives the identity

$$\text{length}(\mathscr{C}) = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r},$$

where the vector field

$$\mathbf{F}(x, y, z) = \langle -y, x, C \rangle.$$

First, calculate length (C).

length (C) $= \int_C 1 ds = \int_{-4\pi}^{4\pi} ||\vec{r}'(t)|| dt$.

Find $\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$, so, $||\vec{r}'(t)|| = \sqrt{\sin^2 t + \cos^2 t + 1}$
$= \sqrt{2}$.

Thus, $\int_C 1 ds = \int_{-4\pi}^{4\pi} \sqrt{2}\, dt = \left[\sqrt{2}t\right]_{-4\pi}^{4\pi} = 8\pi\sqrt{2}$

Now, we calculate $\int_C F \cdot dr = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) dt$.

we know that $F(r(t)) = \langle -\sin(t), \cos(t), C \rangle$.

so, $F(\vec{r}(t)) \cdot \vec{r}'(t) = \langle -\sin(t), \cos(t), C \rangle \cdot \langle -\sin t, \cos t, 1 \rangle$.
$= \sin^2(t) + \cos^2(t) + C = 1 + C$.

so, $\int_C F \cdot dr = \int_{-4\pi}^{4\pi} (1+C)\, dt = (1+C)(8\pi)$.

since, $\int_C F dr = \text{length}(C)$, $8\pi(1+C) = 8\pi\sqrt{2}$.

$1 + C = \sqrt{2} \rightarrow \boxed{C = \sqrt{2} - 1}$.

So when C is equal to $\boxed{\sqrt{2}-1}$, we get the identity that length $(C) = \int_C F \cdot dr$.

2 Question 1 **6 / 6**

- ✓ **+ 6 pts** **Correct**
- **- 2 pts** No explanation (literally only equations and symbols without any context)
- **+ 2 pts** Correct integral for length($$\mathscr{C}$$)
- **+ 2 pts** Correct integral for $$\int_{\mathscr{C}} \mathbf{F} \cdot d \mathbf{r}$$
- **+ 2 pts** Correct final answer
- **+ 1 pts** Small mistake in integral for length($$\mathscr{C}$$)
- **+ 1 pts** Small mistake in integral for Setting up correct integral for $$\int_{\mathscr{C}} \mathbf{F} \cdot d \mathbf{r}$$

2. (7 points) A solid $\mathcal{W}$ occupies the region $x^2+y^2+z^2 \le 25$ and $z \le -\sqrt{x^2+y^2}$, where distance is measured in cm.

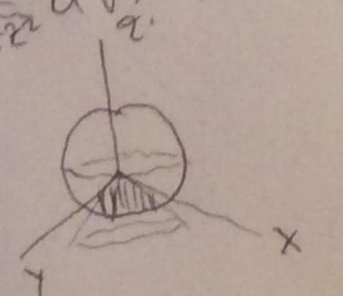
sphere. cone.

The solid has mass density

$$f(x,y,z) = -\frac{z}{\sqrt{x^2+y^2+z^2}} \text{ g cm}^{-3}.$$

Use spherical coordinates to compute the total mass of the solid. in g.

The mass of the solid is given by the integral $\iiint_W \frac{-z}{\sqrt{x^2+y^2+z^2}}\,dV$. We sketch the solid W.



* We find where $z=-\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=25$ intersect using spherical coordinates.

$x^2+y^2+z^2=25 \to \rho=5.$

$z=-\sqrt{x^2+y^2} \to \rho\cos\phi = -\rho\sin\phi.$

Since $\rho \neq 0$, we have that $\tan\phi=-1$, or $\phi = 3\pi/4$ since ϕ must be between 0 and π. Since we want $\tan\phi \le -1$, our theta bounds are between 0 and 2π $(0 \le \theta \le 2\pi)$ and our phi bounds are between $\frac{3\pi}{4}$ and π. $(\frac{3\pi}{4} \le \phi \le \pi)$, Looking at the sketch, $0 \le \rho \le 5$. Thus,

$$\iiint_W \frac{-z}{\sqrt{x^2+y^2+z^2}}\,dV = \int_{3\pi/4}^{\pi}\int_0^{2\pi}\int_0^5 \frac{-\rho\cos\phi}{\rho}\cdot \rho^2\sin\phi\, d\rho\, d\theta\, d\phi.$$

$$= \int_{3\pi/4}^{\pi}\int_0^{2\pi}\int_0^5 -\rho^2\sin\phi\cos\phi\, d\rho\, d\theta\, d\phi = \int_0^{2\pi} d\theta \cdot \int_0^5 -\rho^2\, d\rho \cdot \int_{3\pi/4}^{\pi} \sin\phi\cos\phi\, d\phi.$$

$$= \left[2\pi\right]\cdot\left[-\frac{1}{3}\rho^3\right]_0^5 \cdot \left[\frac{1}{2}\sin^2\phi\right]_{3\pi/4}^{\pi} = 2\pi\left(\frac{-125}{3}\right)\left(\frac{1}{2}\left(0-\frac{1}{2}\right)\right)$$

$$= -\frac{250}{3}\pi \cdot -\frac{1}{4} = \frac{250}{12}\pi = \frac{125}{6}\pi.$$

Thus, the total mass of the solid is $\boxed{\frac{125}{6}\pi \text{ grams.}}$

3 Question 2 **7 / 7**

✓ **- 0 pts** **Correct $$\frac{125\pi}{6} $$**

Bounds for the region in spherical coordinates (3 marks total)

- **- 1 pts** $$\rho$$ bounds incorrect
- **- 1 pts** $$\phi$$ bounds incorrect
- **- 1 pts** $$\theta$$ bounds incorrect

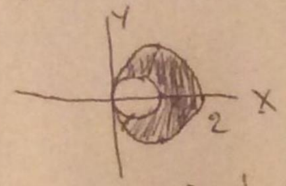
Setting up and solving the integral (4 marks total)

- **- 2 pts** Did not use $$dV=\rho^2\sin(\phi)d\rho d\phi d\theta$$.
- **- 1 pts** Did not correctly convert the integrand into spherical coordinates.
- **- 1 pts** Error in calculating the final integral.

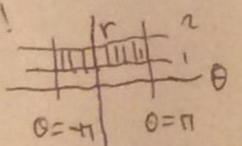
3. (8 points) Let $\mathscr{D}$ be the region bounded between $(x-1)^2+y^2=1$ and $(x-2)^2+y^2=4$. Use the change of variables $(x,y)=(r+r\cos\theta, r\sin\theta)$ to evaluate

$$\iint_{\mathscr{D}} \frac{x^2+y^2}{x}\,dA.$$

First, we sketch D.



Our sketch of D_0 looks like:



Our change of variables is given by $x=r+r\cos\theta$, $y=r\sin\theta$. Calculate the Jacobian $J(G)$.

$$J(G)=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} 1+\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$J(G)= r\cos\theta(1+\cos\theta)+r\sin^2\theta = r+r\cos\theta = x.$

So, $\iint_D \frac{x^2+y^2}{x}dA = \iint_{D_0} \frac{x^2+y^2}{x}\cdot x\,du\,dv = \iint_{D_0}(x^2+y^2)\,du\,dv$, where D_0 is the domain in the r-θ plane that maps onto D through the change of variables. Now, convert both equations to (r,θ).

$(x-1)^2+y^2=1 \to x^2+y^2=2x.$

$(r+r\cos\theta)^2+(r\sin\theta)^2=2r(1+\cos\theta).$

$2r^2(1+\cos\theta)=2r(1+\cos\theta)$. $\to$ the solutions to this can be $r=0, r=1$, or $1+\cos\theta=0$, but for D_0 to be one-to-one on its interior, let us choose this to be $\underline{r=1}$.

Similarly, $(x-2)^2+y^2=4$ is $x^2+y^2=4x$.

so $2r^2(1+\cos\theta)=4r(1+\cos\theta)$. $\to$ we choose this to be $r=2$ in D_0.

Thus, $\underline{1\le r\le 2}$. We see that as we move θ from $-\pi$ to π, the entire region D in the xy-plane, and region D_0 maps onto the entire region D in the xy-plane, and is one-on-one on its interior. So, $\underline{-\pi\le\theta\le\pi}$. Thus,

$\iint_D \frac{x^2+y^2}{x}dA = \iint_{D_0}(x^2+y^2)\,du\,dv = \iint_{D_0} 2r^2(1+\cos\theta)\,dr\,d\theta.$

$= \int_{-\pi}^{\pi}\int_1^2 2r^2(1+\cos\theta)\,dr\,d\theta.$

$= \int_{-\pi}^{\pi}(1+\cos\theta)\,d\theta \cdot \int_1^2 2r^2\,dr = \left[\theta+\sin\theta\right]_{-\pi}^{\pi}\cdot\left[\frac{2}{3}r^3\right]_1^2$

$= 2\pi\cdot\frac{14}{3} = \boxed{28\pi/3}$

4 Question 3 **8 / 8**

- ✓ **+ 8 pts** **Everything correct**
- **+ 2 pts** Correct bounds for r: 1 to 2
- **+ 2 pts** Correct bounds for theta: 0 to 2pi
- **+ 2 pts** Correct jacobian r+r*cos(theta)
- **+ 1 pts** Correct integrand 2r
- **+ 1 pts** Correct final answer 28pi/3

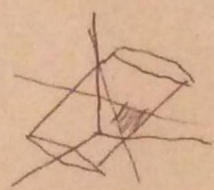
4. (9 points) Let S be the part of the cylinder $y^2+z^2=1$ bounded between $z=0$, $z=1+x$ and $z=1-x$, oriented with the downward pointing normal.

Find the flux of the vector field

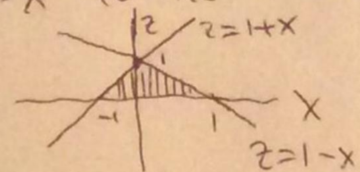
$$\mathbf{F}(x,y,z)=\langle 0,\, y,\, z\rangle$$

across S.

The flux of the vector field across S is $\iint_S \vec{F}\cdot d\vec{S}$. We sketch the surface S.



Also, we sketch projections of the curves $z=0$, $z=1+x$, and $z=1-x$ to the xz-plane.



We parametrize the cylinder as $G(\theta,x)=(x,\cos\theta,\sin\theta)$. The ranges of x and z from the sketch are $\{0\le z\le 1,\ z-1\le x\le 1-z\}$. so, $0\le\sin\theta\le 1$, and $\sin\theta-1\le x\le 1-\sin\theta$. Since θ must be between 0 and π, we have $\underline{0\le\theta\le\pi}$ and $\underline{\sin\theta-1\le x\le 1-\sin\theta}$. We find the tangent vectors.

$T_\theta=(0,-\sin\theta,\cos\theta)$ $\rightarrow$ $T_\theta\times T_x=(0,\cos\theta,\sin\theta)$.

$T_x=(1,0,0)$ we want the downward-pointing normal,

so $N=T_x\times T_\theta=(0,-\cos\theta,-\sin\theta)$.

$F(G(\theta,x))=\langle 0,\cos\theta,\sin\theta\rangle$

so $F\cdot N=\langle 0,\cos\theta,\sin\theta\rangle\cdot\langle 0,-\cos\theta,-\sin\theta\rangle=-\cos^2\theta-\sin^2\theta=-1$.

we see that $\iint_S\vec{F}\cdot d\vec{S}=\iint_D\vec{F}\cdot\vec{N}\,du\,dv=\int_0^\pi\int_{\sin\theta-1}^{1-\sin\theta}(-1)\,dx\,d\theta$.

$=\int_0^\pi(\sin\theta-1)-(1-\sin\theta)\,d\theta=\int_0^\pi(2\sin\theta-2)\,d\theta$

$=\left[-2\cos\theta-2\theta\right]_0^\pi=(2-2\pi)-(-2)=4-2\pi$.

The flux through S is $\boxed{4-2\pi.}$

5 Question 4 **9 / 9**

✓ + **1 pts** **Attempts to parameterize $$S$$**

✓ + **2 pts** **Correct choice of $$G(x,\theta) = (x,\cos\theta,\sin\theta)$$ _(or equivalent)_**

✓ + **1 pts** **Correct limits for $$\theta$$:**

*** Either $$0\leq \theta\leq \pi$$ for top half**

*** Or $$\pi\leq \theta\leq 2\pi$$ for bottom half**

(or equivalent)

✓ + **1 pts** **Correct limits for $$x$$:**

$$\sin\theta-1\leq x\leq 1-\sin\theta$$ _(or equivalent)_

✓ + **2 pts** **Correctly computes (or cites) normal vector:**

*** Either $$\mathbf N = \langle0,-\cos\theta,-\sin\theta\rangle$$ for top half**

*** Or $$\mathbf N = \langle0,\cos\theta,\sin\theta\rangle$$ for bottom half**

(or equivalent)

+ **1 pts** _(Partial credit)_ Error in normal computation or correctly computes normal, but orientation incorrect.

✓ + **2 pts** **Correctly computes the flux to be:**

*** Either $$4-2\pi$$ for the top half**

*** Or $$4+2\pi$$ for the bottom half**

(Credit awarded if errors correctly carried)

+ **1 pts** _(Partial credit)_ Minor error in computation or added superfluous flux integrals to answer.

+ **0 pts** No credit due.

+ **1 pts** _(Partial credit)_ Computes normal of incorrect parameterization.