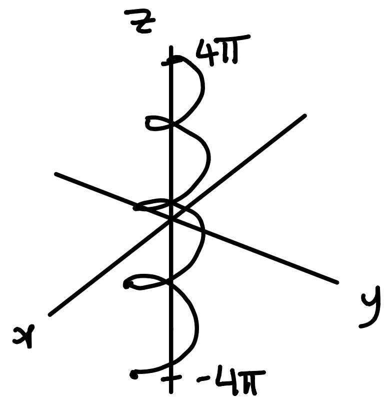
1. (6 points) Let $\mathscr{C}$ be the curve with parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ for $-4\pi \le t \le 4\pi$. Find the value of the constant C that gives the identity

$$\text{length}(\mathscr{C}) = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r},$$

where the vector field

$$\mathbf{F}(x, y, z) = \big\langle -y\,,\, x\,,\, C \big\rangle.$$



① Find $\vec{r}'(t)$ and $\|\vec{r}'(t)\|$

$\vec{r}'(t) = \langle -\sin t, \cos t, t \rangle$

$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + t^2} = \sqrt{1 + t^2}$

② Find length(C)

$$\text{length}(C) = \int_C 1\, ds = \int_C \|\vec{r}'(t)\|\, dt = \int_{-4\pi}^{4\pi} \sqrt{1+t^2}\, dt = 161.64$$

③ $\int_C F \cdot dr = \int_C F(r(t)) \cdot \vec{r}'(t)\, dt$

$F(r(t)) = \langle -\sin t, \cos t, C \rangle$

$$\int_{-4\pi}^{4\pi} \langle -\sin t, \cos t, C \rangle \cdot \langle -\sin t, \cos t, t \rangle\, dt$$

$$= \int_{-4\pi}^{4\pi} \sin^2 t + \cos^2 t + Ct\, dt = \int_{-4\pi}^{4\pi} 1 + Ct\, dt$$

2. (7 points) A solid $\mathscr{W}$ occupies the region $x^2+y^2+z^2 \leq 25$ and $z \leq -\sqrt{x^2+y^2}$, where distance is measured in cm.

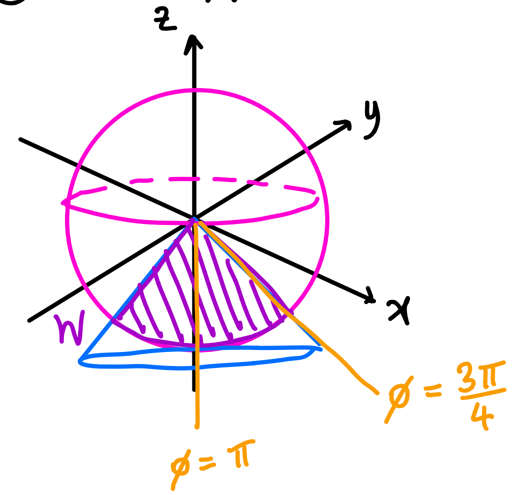
   The solid has mass density

$$f(x,y,z) = -\frac{z}{\sqrt{x^2+y^2+z^2}} \quad \mathrm{g\,cm}^{-3}.$$

   Use spherical coordinates to compute the total mass of the solid.

Total mass: $\iiint_W f(x,y,z)\,dW$

① Sketch W



② Determine bounds

$$W' = \left\{ \begin{array}{l} 0 \leq \rho \leq 5 \\ 0 \leq \theta \leq 2\pi \\ \frac{3\pi}{4} \leq \phi \leq \pi \end{array} \right\}$$

③ Find $f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

$$f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) = \frac{-\rho\cos\phi}{\rho} = -\cos\phi$$

④ Compute $\iiint_W f(x,y,z)\,dW$

$$\iiint_W f(x,y,z)\,dz = \iiint_{W'} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \cdot \rho^2\sin\phi\,dW'$$

$$= \int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi}\int_0^5 -\cos\phi \cdot \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_{\frac{3\pi}{4}}^{\pi} -\sin\phi\cos\phi \, d\phi \int_0^5 \rho^2 \, d\rho$$

$$= \cancel{2}\pi \int_{3\pi/4}^{\pi} -\frac{\sin 2\phi}{\cancel{2}} \, d\phi \left[\frac{\rho^3}{3} \right]_0^5$$

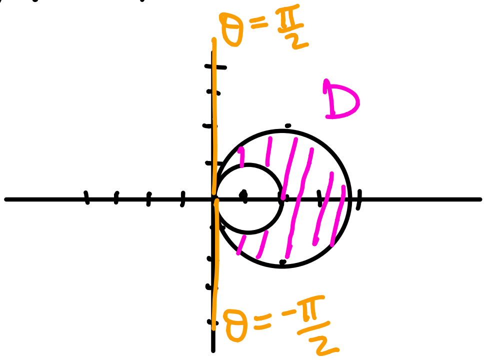
$$= -\pi\left(\frac{5^3}{3}\right) \int_{3\pi/4}^{\pi} \sin 2\phi \, d\phi = -\pi\left(\frac{5^3}{3}\right) \left[-\frac{\cos 2\phi}{2} \right]_{3\pi/4}^{\pi}$$

$$= -\pi\left(\frac{5^3}{3}\right)\left(-\frac{1}{2} + 0\right) = \frac{5^3}{6}\pi = \boxed{\frac{125}{6}\pi \ g}$$

3. (8 points) Let $\mathscr{D}$ be the region bounded between $(x-1)^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 4$. Use the change of variables $(x, y) = (r + r\cos\theta, r\sin\theta)$ to evaluate

$$\iint_{\mathscr{D}} \frac{x^2+y^2}{x}\, dA.$$

① First, Sketch D.



② Then, Determine bounds of the transformed region.

$(x,y) = G(r,\theta) = (r + r\cos\theta, r\sin\theta)$

The lower limit is $(x-1)^2 + y^2 = 1$. Transformed, this gives us $(r + r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$. Expand this equation to solve for r:

$$r^2 + r^2\cos\theta - r + r^2\cos\theta + r^2\cos^2\theta - r\cos\theta - r - r\cos\theta + \cancel{1} + r^2\sin^2\theta = \cancel{1}$$

$$r^2 + 2r^2\cos\theta - 2r + r^2 - 2r\cos\theta = 0$$

$$2r^2 + 2r^2\cos\theta - 2r - 2r\cos\theta = 0$$

$$2r(r + r\cos\theta - 1 - \cos\theta) = 0$$

$2r = 0$

$\therefore r = 0$

$r + r\cos\theta - 1 - \cos\theta = 0$

$r(1+\cos\theta) = 1 + \cos\theta$

$r = \dfrac{1+\cos\theta}{1+\cos\theta} = 1 \quad \therefore r = 1$

The upper limit is $(x-2)^2 + y^2 = 4$. Transformed, this gives us $(r + r\cos\theta - 2)^2 + r^2\sin^2\theta = 4$. Expand this equation to solve for r:

$$r^2 + r^2\cos\theta - 2r + r^2\cos\theta + r^2\cos^2\theta - 2r\cos\theta - 2r - 2r\cos\theta + \cancel{4} + r^2\sin^2\theta = \cancel{4}$$

$$r^2 + 2r^2\cos\theta - 4r + r^2 - 4r\cos\theta = 0$$

$$2r^2 + 2r^2\cos\theta - 4r - 4r\cos\theta = 0$$

$$2r(r + r\cos\theta - 2 - 2\cos\theta) = 0$$

$$2r = 0 \qquad r + r\cos\theta - 2 - 2\cos\theta = 0$$

$$r = 0 \qquad r(1+\cos\theta) = 2 + 2\cos\theta$$

$$r = \frac{2+2\cos\theta}{1+\cos\theta} = \frac{2(1+\cos\theta)}{1+\cos\theta} = 2 \quad \therefore r = 2$$

Applying the transformation to the bounds of θ, we get that $1 \le r \le 2$.

To find the bounds of θ, we observe that $x \ge 0$ for all of the region D. This implies that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Thus, the bounds of D_0 are: $D_0 = \left\{1 \le r \le 2,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$

③ Calculate the Jacobian.

$$(x,y) = G(r,\theta) = (r + r\cos\theta, r\sin\theta)$$

$$\mathrm{Jac}(G) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} = 1+\cos\theta & \frac{\partial x}{\partial \theta} = -r\sin\theta \\ \frac{\partial y}{\partial r} = \sin\theta & \frac{\partial y}{\partial \theta} = r\cos\theta \end{vmatrix}$$

$$= r\cos\theta(1+\cos\theta) + r\sin^2\theta$$

$$= r\cos\theta + r\cos^2\theta + r\sin^2\theta = r + r\cos\theta$$

④ Find $f(r + r\cos\theta, r\sin\theta)$

$$f(x,y) = \frac{x^2+y^2}{x} \quad \therefore f(r+r\cos\theta, r\sin\theta) = \frac{(r+r\cos\theta)^2 + r^2\sin^2\theta}{r+r\cos\theta}$$

$$= \frac{r^2 + 2r\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}{r + r\cos\theta} = \frac{r^2 + 2r\cos\theta + r^2}{r + r\cos\theta} = \frac{2r^2 + 2r\cos\theta}{r + r\cos\theta}$$

⑤ compute the integral using the transformation.

$$\iint_D f(x,y)\,dA = \iint_{D_0} f(r+r\cos\theta, r\sin\theta)\,|\text{Jac}\,G|\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_1^2 \frac{2r^2+2r\cos\theta}{\cancel{r+r\cos\theta}} \cdot \cancel{r+r\cos\theta}\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_1^2 2r^2+2r\cos\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{2}{3}r^3 + r^2\cos\theta\right]_1^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{16}{3} + 4\cos\theta - \frac{2}{3} - \cos\theta\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{14}{3} + 3\cos\theta\,d\theta$$

$$= \left[\frac{14}{3}\theta + 3\sin\theta\right]_{-\pi/2}^{\pi/2} = \frac{14\pi}{6} + 3 + \frac{14\pi}{6} + 3 = \boxed{\frac{14\pi}{3} + 6}$$

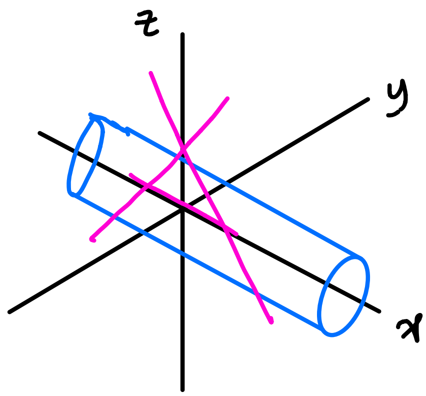
4. (9 points) Let S be the part of the cylinder $y^2 + z^2 = 1$ bounded between $z = 0$, $z = 1 + x$ and $z = 1 - x$, oriented with the downward pointing normal.

   Find the flux of the vector field
   $$\mathbf{F}(x, y, z) = \langle 0\,,\, y,\, z \rangle$$
   across S.

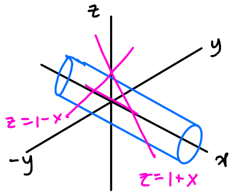
$z = \sqrt{1-y^2}$

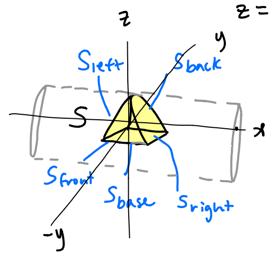
① Sketch S

$x =$



① Sketch S





$z = \sqrt{1-y^2}$

$x^2+y^2=1$

$x=0 \quad x=1+z$

$x=1-z$

$z=x-1$

$z=1-x$

② Observe that $S = S_{base} + S_{right} + S_{left} + S_{front} + S_{back}$.
Parametrize each surface separately.

③ Parametrize S_{base}. Observe that $z=0$ for all S_{base}. So:

$y^2+0^2=1 \quad \therefore y=\pm 1$

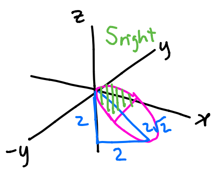
$0=1+x, \ 0=1-x \quad \therefore x=\pm 1$

Therefore S_{base} is the unit square.

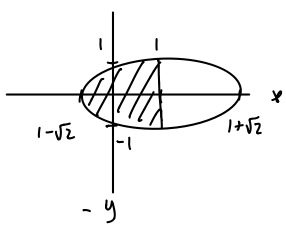
We can now parametrize it as:

$S_{base} = G_1(x,y) = (x,y,0)$ for $0 \le x \le 1$ and $-1 \le y \le 1$

④ Parametrize S_{right}. Based on the sketch of S_{right} below,



S_{right} is the upper half of an ellipse with vertical radius of $\sqrt{2}$ and horizontal radius of 1. Its z coordinate goes from 0 to 1.



$$\frac{(x-1)^2}{\sqrt{2}} + y^2 = 1$$

$$(x-1)^2 + \sqrt{2}y^2 = \sqrt{2}$$

$$(x-1)^2 = \sqrt{2} - \sqrt{2}y^2$$

$$x-1 = -\sqrt{\sqrt{2}-\sqrt{2}y^2}$$

$$x = 1 - \sqrt{\sqrt{2}-\sqrt{2}y^2}$$

$$\therefore x = 1 - 2^{1/4}\sqrt{1-y^2}$$

Thus, we can parametrize it as:

$S_{right} = G_2(y,z) = (1-2^{1/4}\sqrt{1-y^2}, y, z)$ for $-1 \le y \le 1$, $0 \le z \le 1$

⑤ Parametrize S_{left}.