Math 32B Midterm 2D

YAN QU

TOTAL POINTS

40 / 40

QUESTION 1

1 Green's Theorem 6 / 6

✓ + **2 pts** Correct application of Green's Theorem

✓ + **1 pts** Correct computation of $\mathrm{curl}_z\mathbf{F} = -2-2y$

✓ + **1 pts** Correct limits for the rectangle (must have all four correct to receive credit)

✓ + **1 pts** Correct answer of -16 (requires correct integrand and limits to receive credit)

✓ + **1 pts** Solution clearly explained (and at the very least should mention Green's Theorem)

+ **0 pts** No credit due
+ **1 pts** Applied Green's Theorem correctly but with wrong orientation. (Partial credit)

QUESTION 2

2 Change of variables 14 / 14

✓ + **2 pts** Linear change of variables

✓ + **3 pts** Appropriate linear change of variables

✓ + **2 pts** Correct (u,v) region

✓ + **1 pts** Correct Jacobian

✓ + **1 pts** Use Jacobian

✓ + **2 pts** Correctly substitute u and v in δ/integrand.

✓ + **1 pts** Calculate correctly

✓ + **1 pts** Clear and organized solution, units

✓ + **1 pts** Accurate diagram, or accurate description of (x,y) region

+ **1 pts** Partial credit for error in finding (u,v) region, Jacobian, or δ(x(u,v),y(u,v))
+ **0 pts** No credit due
+ **1 pts** Sanity check: recognize that a negative answer is incorrect (does not apply if "corrected" by taking absolute value)

QUESTION 3

3 Surface integral 12 / 12

✓ + **4 pts** Correct parametrization and parameter domain

+ **2 pts** Partial credits for parametrization

✓ + **4 pts** Correct tangent vector and normal vector calculation

+ **2 pts** Partial credits for tangent and normal

✓ + **4 pts** Correct double integral calculation

+ **2 pts** Partial credits for double integral
- **1 pts** You are almost there.
+ **1 pts** Almost nothing correct.
+ **0 pts** Nothing correct

QUESTION 4

4 Integration by parts 8 / 8

✓ + **1 pts** Clear explanation

✓ + **3 pts** (a) correct

✓ + **4 pts** (b) correct

+ **0 pts** Incorrect
+ **2 pts** (a) incomplete argument, but right idea
+ **2 pts** (b) incomplete argument, but right idea
+ **2 pts** (a) slight error
+ **3 pts** (b) slight error/unfinished
+ **1 pts** (a) started correctly, e.g. wrote integration by parts formula
+ **1 pts** (b) started correctly

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 2
2/22/2019

Name: Yan Qu
SID: 605206804
TA Section: 4A

Time Limit: 50 Minutes

Version (↓)

This exam contains 12 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

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Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\, dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\, dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\, dA$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\, dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\, dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\, dA$

Probability formulas

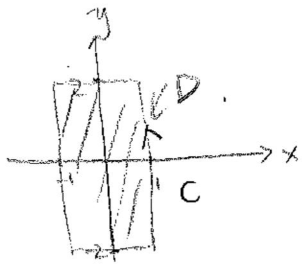
- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\, dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\, dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\, p_X(x)\, dx.$
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\, dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\, dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, p_{X,Y}(x,y)\, dxdy$

1. (6 points) Let C be the boundary of the rectangle $\mathcal{D} = \{-1 \le x \le 1\,,\ -2 \le y \le 2\}$ oriented counterclockwise and let

$$\mathbf{F}(x,y) = \left\langle e^{x^8} + y^2\,,\ \sin(y^4) - 2x \right\rangle.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$



By green's theorem, $\oint_C F \cdot dr = \iint_D \operatorname{curl}_z \vec{F}\, dA$ $\because$ C is oriented counterclockwise, the correct orientation

$\because F_1 = e^{x^8} + y^2 \quad F_2 = \sin(y^4) - 2x$

$$\operatorname{Curl}_z \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= -2 - 2y = -2(1+y)$$

$$\oint_C F \cdot dr = -2\int_{-1}^{1}\int_{-2}^{2} (1+y)\, dy\, dx$$

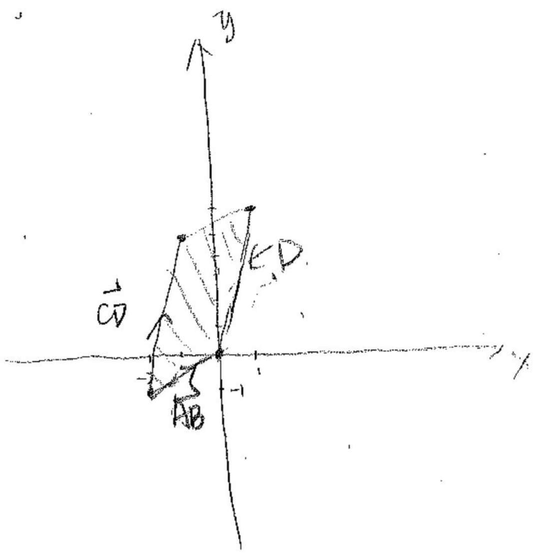
$$= -2\int_{-1}^{1} \left[\frac{(1+y)^2}{2} \right]_{-2}^{2} dx$$

$$= -2\int_{-1}^{1} \frac{9}{2} - \frac{1}{2}\, dx$$

$$= -2\,[4x]_{-1}^{1}$$

$$= -16$$

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-2,-1),(0,0),(1,6),(-1,5)$ (where distance is measured in meters) and with mass density $\delta(x,y) = (2y-x)\ \mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$.



The mass is

$$M = \iint_D \delta(x,y)\,dA$$

Change the variables to u, v.

Let u be the distance away in $\vec{AB}$ direction and v be the distance away in $\vec{CD}$ direction

$\therefore$ all (x,y) in D can be written as

$$u\vec{AB} + v\vec{CD} \qquad \vec{AB} = \langle A, B\rangle \quad \vec{CD} = \langle C, D\rangle$$

$$\begin{cases} x = uA + vC \\ y = Bu + Dv \end{cases}$$

in this case $\vec{AB} = \langle 2, 1\rangle$ $\vec{CD} = \langle 1, 6\rangle$

$$\therefore \begin{aligned} x &= 2u + v \\ y &= u + 6v \end{aligned} \Rightarrow \begin{cases} v = \dfrac{x-2y}{-11} \\ u = \dfrac{6x-y}{11} \end{cases}$$

$$\therefore dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right| = \left|\begin{matrix} 2 & 1 \\ 1 & 6 \end{matrix}\right| = 11$$

$\because$ $(-2,-1)$ is when u and v are minimum and $(1,6)$ is when u, v are maximum

the bounds are $v_{min} = \dfrac{-2-2(-1)}{-11} = 0$ $u_{min} = \dfrac{6(-2)-(-1)}{11} = -1$

$v_{max} = \dfrac{1-2(6)}{-11} = 1$ $u_{max} = \dfrac{6(1)-6}{11} = 0$

$$\therefore \ M = \int_0^1 \int_{-1}^0 (\cancel{2u} + 12v - \cancel{2u} - v)\, 11 \, du \, dv$$

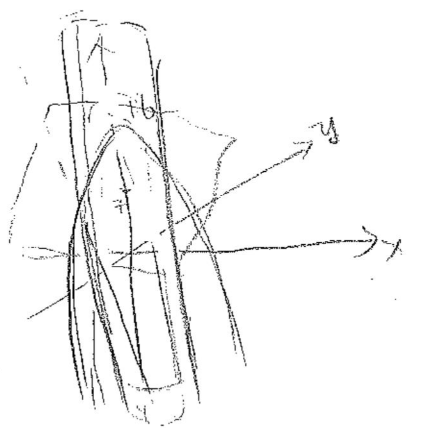
$$= 11^2 \int_{-1}^0 \int_0^1 11 v \, dv \, du$$

$$= 11^2 \int_{-1}^0 \left[\frac{v^2}{2} \right]_0^1 du$$

$$= 11^2 \left[\frac{1}{2} u \right]_{-1}^0$$

$$= \frac{11^2}{2} \text{ kg}$$

3. (12 points) Find the area of the part of the paraboloid $z = 16 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 4$ and above the plane $z = 7$.



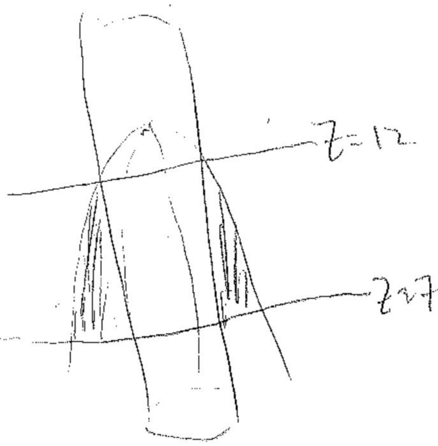
$z = 16 - x^2 - y^2 \quad x^2 + y^2 = 16 - z$

$16 - z = 4 \quad z = 12$

Let the surface be parameterized by

$\langle x, y, 16 - x^2 - y^2 \rangle$

$\because$ the area $= \iint_S 1 \, dS$



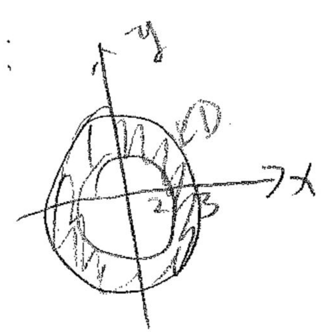
$\because$ the part is above the plane $7 \geq z$

$7 \geq 16 - x^2 - y^2 \quad x^2 + y^2 \leq 9$

$\because$ outside the cylinder $\to$ below $z = 12$

$\therefore x^2 + y^2 \geq 4$

$\iint_S dS = \iint_D \left\| \frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} \right\| dx\,dy$



$\frac{\partial G}{\partial x} = \langle 1, 0, -2x \rangle$

$\frac{\partial G}{\partial y} = \langle 0, 1, -2y \rangle$

$\frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle$

$\left\| \frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} \right\| = \sqrt{(2x)^2 + (2y)^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$

$$\iint_S dS = \iint_D \sqrt{4x^2+4y^2+1}\, dx\, dy$$

change to polar coordinates

$$\iint_S dS = \int_0^{2\pi}\int_2^3 \sqrt{4r^2+1}\; r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{12}(4r^2+1)^{\frac{3}{2}}\right]_2^3 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{12}\cdot 37^{\frac{3}{2}} - \frac{1}{12}\cdot 17^{\frac{3}{2}}\right) d\theta$$

$$= \frac{1}{12}\left(37^{\frac{3}{2}} - 17^{\frac{3}{2}}\right) 2\pi$$

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve $\mathcal{C}$ oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

   (a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x, y)$, $g(x, y)$ we have the identity

$$\iint_{\mathcal{D}} f \Delta g \, dA - \iint_{\mathcal{D}} g \Delta f \, dA = \oint_{\mathcal{C}} f \nabla g \cdot \mathbf{n} \, ds - \oint_{\mathcal{C}} g \nabla f \cdot \mathbf{n} \, ds.$$

(*Hint: Recall that* $\Delta f = \operatorname{div} \nabla f$)

$$\because \iint_D f \operatorname{div} \vec{F} \, dA = \oint_C f \vec{F} \cdot \hat{n} \, ds - \iint_D \nabla f \vec{F} \, dA$$

$$\iint_D f \Delta g \, dA = \iint_D f \operatorname{div} \nabla g \, dA$$

$$= \oint_C f \nabla g \hat{n} \, ds - \iint_D \nabla f \nabla g \, dA$$

$$\iint_D g \Delta f \, dA = \iint_D g \operatorname{div} \nabla f \, dA$$

$$= \oint_C g \nabla f \hat{n} \, ds - \iint_D \nabla g \nabla f \, dA$$

$$\therefore \iint_D f \Delta g \, dA - \iint_D g \Delta f \, dA$$

$$= \oint_C f \nabla g \hat{n} \, ds - \oint_C g \nabla f \hat{n} \, ds.$$

(b) Suppose that $f(x, y)$, $g(x, y)$ are smooth, non-zero, scalar functions satisfying the equations

$$\Delta f = \lambda f \quad \text{for all } (x, y) \in \mathcal{D},$$
$$\Delta g = \mu g \quad \text{for all } (x, y) \in \mathcal{D},$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x, y)$, $g(x, y)$ satisfy the boundary condition

$$f(x, y) = 0 \quad \text{for all } (x, y) \in \mathcal{C},$$
$$g(x, y) = 0 \quad \text{for all } (x, y) \in \mathcal{C}.$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x, y) g(x, y)\, dA = 0.$$

$\because f(x,y) = g(x,y) = 0$ for all $(x,y) \in C$

$\oint_C f \nabla g \cdot \vec{n}\, ds = \oint_C g \nabla f \cdot \vec{n}\, ds = 0$

$\therefore \iint_D f \Delta g\, dA - \iint_D g \Delta f\, dA = 0$

$\iint_D f \Delta g\, dA = \iint_D g \Delta f\, dA$

$\because \Delta f = \lambda f$, $\Delta g = \mu g$ for all $(x,y) \in D$

$\iint_D f \mu g\, dA = \iint_D g \lambda f\, dA$

$\Rightarrow (\mu - \lambda) \iint_D gf\, dA = 0$ ①

$\because \lambda \neq \mu \quad \therefore \mu - \lambda \neq 0$

$\therefore$ the only way for ① to be true

is $\iint_D fg\, dA = \iint_D gf\, dA = 0$