Math 32B Midterm 2D

MINGYANG ZHANG

TOTAL POINTS

38 / 40

QUESTION 1

1 Green's Theorem 6 / 6

✓ + **2 pts** Correct application of Green's Theorem

✓ + **1 pts** Correct computation of $$\mathrm{curl}_z\mathbf{F} = -2-2y$$

✓ + **1 pts** Correct limits for the rectangle (must have all four correct to receive credit)

✓ + **1 pts** Correct answer of $$-16$$ (requires correct integrand and limits to receive credit)

✓ + **1 pts** Solution clearly explained (and at the very least should mention Green's Theorem)

+ **0 pts** No credit due

+ **1 pts** Applied Green's Theorem correctly but with wrong orientation. (Partial credit)

QUESTION 2

2 Change of variables 14 / 14

✓ + **2 pts** Linear change of variables

✓ + **3 pts** Appropriate linear change of variables

✓ + **2 pts** Correct (u,v) region

✓ + **1 pts** Correct Jacobian

✓ + **1 pts** Use Jacobian

✓ + **2 pts** Correctly substitute u and v in δ/integrand.

✓ + **1 pts** Calculate correctly

✓ + **1 pts** Clear and organized solution, units

✓ + **1 pts** Accurate diagram, or accurate description of (x,y) region

+ **1 pts** Partial credit for error in finding (u,v) region, Jacobian, or δ(x(u,v),y(u,v))

+ **0 pts** No credit due

+ **1 pts** Sanity check: recognize that a negative answer is incorrect (does not apply if "corrected" by taking absolute value)

QUESTION 3

3 Surface integral 10 / 12

+ **4 pts** Correct parametrization and parameter domain

✓ + **2 pts** Partial credits for parametrization

✓ + **4 pts** Correct tangent vector and normal vector calculation

+ **2 pts** Partial credits for tangent and normal

✓ + **4 pts** Correct double integral calculation

+ **2 pts** Partial credits for double integral

- **1 pts** You are almost there.

+ **1 pts** Almost nothing correct.

+ **0 pts** Nothing correct

QUESTION 4

4 Integration by parts 8 / 8

✓ + **1 pts** Clear explanation

✓ + **3 pts** (a) correct

✓ + **4 pts** (b) correct

+ **0 pts** Incorrect

+ **2 pts** (a) incomplete argument, but right idea

+ **2 pts** (b) incomplete argument, but right idea

+ **2 pts** (a) slight error

+ **3 pts** (b) slight error/unfinished

+ **1 pts** (a) started correctly, e.g. wrote integration by parts formula

+ **1 pts** (b) started correctly

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 2
2/22/2019

Name: Mingyang Zhang
SID: 405170429
TA Section: 3E

Time Limit: 50 Minutes

Version ⓓ

This exam contains 12 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

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Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x, y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$

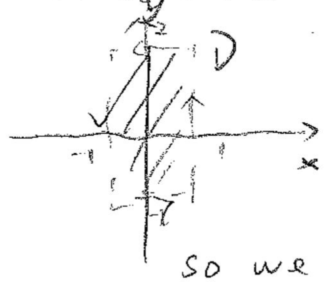
1. (6 points) Let C be the boundary of the rectangle $\mathcal{D} = \{-1 \le x \le 1\,,\ -2 \le y \le 2\}$ oriented counterclockwise and let

$$\mathbf{F}(x,y) = \left\langle e^{x^8} + y^2\,,\ \sin(y^4) - 2x \right\rangle.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

First, draw the Graph of D.



by Green's Theorem,

we have that $\oint_C \vec{F}\,d\vec{r} = \iint_D \text{curl}_z(\vec{F})\,dA$

so we compute $\text{curl}_z(\vec{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

$= -2 - 2y$

since C oriented counterclockwise

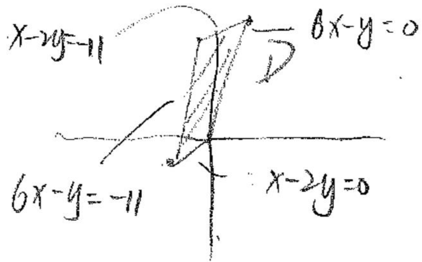
$$\oint_C \vec{F}\,d\vec{r} = \int_{-2}^{2}\int_{-1}^{1} -2 - 2y\,dx\,dy = \int_{-2}^{2} -2x - 2xy \Big|_{-1}^{1} dy$$

$$= \int_{-2}^{2} -2 - 2y - 2 - 2y\,dy = \int_{-2}^{2} -4 - 4y\,dy$$

$$= -4y - 2y^2 \Big|_{-2}^{2} = -8 - 8 - (8 - 8) = -16$$

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-2,-1),(0,0),(1,6),(-1,5)$ (where distance is measured in meters) and with mass density $\delta(x,y)=(2y-x)$ $\mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$.

First, draw the Graph of D



so let $u=6x-y$ s.t. $u\in[-11,0]$

$v=x-2y$ $v\in[-11,0]$

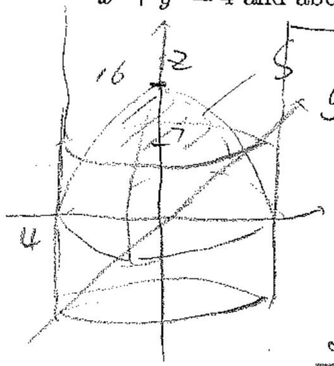
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}\right| = \left|\frac{1}{\det\begin{bmatrix}6 & -1\\ 1 & -2\end{bmatrix}}\right| = \frac{1}{11}$$

so. total mass of D $=\int_D \frac{1}{11}\delta(x,y)\,dA$

$$=\int_{-11}^{0}\int_{-11}^{0}\frac{-v}{11}\,du\,dv$$

$$=\int_{-11}^{0}-v\,dv=\frac{121}{2}=60.5\text{ kg}$$

3. (12 points) Find the area of the part of the paraboloid $z = 16 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 4$ and above the plane $z = 7$.



first draw the Graph

then parametrize the paraboloid as $\vec{G}(r,\theta) = \langle r\cos\theta, r\sin\theta, 16 - r^2 \rangle$, $r \in [0, 3]$, $\theta \in [0, 2\pi]$

since when $z = 7$, $r^2 = 16 - 7 = 9$ $\Rightarrow r = 3$

$\frac{\partial G}{\partial r} = \langle \cos\theta, \sin\theta, -2r \rangle$

$\frac{\partial G}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\|\vec{n}\| = \frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$

$\|\vec{n}\| = \sqrt{4r^4 + r^2}$

$\text{Area}(S) = \int_0^3 \int_0^{2\pi} \sqrt{4r^4 + r^2}\, d\theta\, dr = 2\pi \int_0^{\sqrt{7}} r\sqrt{4r^2+1}\, dr$

let $u = 4r^2 + 1$ $du = 8r\,dr$ $\frac{du}{8} = r\,dr$

$\Rightarrow \text{Area}(S) = \frac{2\pi}{4} \cdot \frac{2}{3} \left[(4r^2+1)^{\frac{3}{2}} \right]_0^{\sqrt{7}} = \frac{\pi}{6} (37^{\frac{3}{2}} - 1)$

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve $\mathcal{C}$ oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

(a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x,y)$, $g(x,y)$ we have the identity

$$\iint_{\mathcal{D}} f\Delta g\, dA - \iint_{\mathcal{D}} g\Delta f\, dA = \oint_{\mathcal{C}} f\nabla g \cdot \mathbf{n}\, ds - \oint_{\mathcal{C}} g\nabla f \cdot \mathbf{n}\, ds.$$

(*Hint: Recall that* $\Delta f = \operatorname{div} \nabla f$)

From the formula of integration by parts

we have then.

$$\iint_D f \Delta g\, dA = \oint_C f \nabla g \cdot \hat{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA \quad ①$$

we also have

$$\iint_D g \Delta f\, dA = \oint_C g \nabla f \cdot \hat{n}\, ds - \iint_D \nabla g \nabla f\, dA \quad ②$$

① − ② : $\iint_D f \Delta g\, dA - \iint_D g \Delta f\, dA = \oint_C f \nabla g \cdot \hat{n}\, ds - \oint_C g \nabla f \cdot \hat{n}\, ds$

(b) Suppose that $f(x,y)$, $g(x,y)$ are smooth, non-zero, scalar functions satisfying the equations

$$\Delta f = \lambda f \quad \text{for all } (x,y) \in \mathcal{D},$$
$$\Delta g = \mu g \quad \text{for all } (x,y) \in \mathcal{D},$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x,y)$, $g(x,y)$ satisfy the boundary condition

$$f(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C},$$
$$g(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C}.$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x,y)g(x,y)\,dA = 0.$$

from part (a). and $\Delta f = \lambda f$ $\Delta g = \mu g$

we have.

$\mu \iint_D fg\, dA - \lambda \iint_D fg\, dA = \oint_C f \nabla g \cdot \hat{n}\, ds - \oint_C g \nabla f \cdot \hat{n}\, ds$

since $f(x,y) = 0$ $g(x,y) = 0$ for all $(x,y) \in C$.

$\Rightarrow$ $\oint_C f \nabla g \hat{n}\, ds = 0 = \oint_C g \nabla f \cdot \hat{n}\, ds$

$\Rightarrow (\mu - \lambda) \iint_D fg\, dA = 0$

Since $\mu \neq \lambda$. so $\iint_D f(x,y)\, g(x,y)\, dA = 0$