Math 32B Midterm 2U

NIKIL SELVAM

TOTAL POINTS

40 / 40

QUESTION 1

1 Green's Theorem 6 / 6

✓ + **2 pts** **Correct application of Green's Theorem**

✓ + **1 pts** **Correct computation of $$\mathrm{curl}_z\mathbf{F} = 6xy+4$$**

✓ + **1 pts** **Correct limits for the rectangle (must have all four correct to receive credit)**

✓ + **1 pts** **Correct answer of $$32$$ (requires correct integrand and limits to receive credit)**

✓ + **1 pts** **Solution clearly explained (and at the very least should mention Green's Theorem)**

+ **0 pts** No credit due
+ **1 pts** Applied Green's Theorem correctly but with wrong orientation. (Partial credit)

QUESTION 2

2 Change of variables 14 / 14

✓ + **2 pts** **Linear change of variables**

✓ + **3 pts** **Appropriate linear change of variables**

✓ + **2 pts** **Correct (u,v) region**

✓ + **1 pts** **Correct Jacobian**

✓ + **1 pts** **Use Jacobian**

✓ + **2 pts** **Correctly substitute u and v in δ/integrand.**

✓ + **1 pts** **Calculate correctly**

✓ + **1 pts** **Clear and organized solution, units**

✓ + **1 pts** **Accurate diagram, or accurate description of (x,y) region**

+ **1 pts** Partial credit for error in finding (u,v) region, Jacobian, or δ(x(u,v),y(u,v))
+ **0 pts** No credit due
+ **1 pts** Sanity check: recognize that a negative answer is incorrect (does not apply if "corrected" by taking absolute value)

QUESTION 3

3 Surface integral 12 / 12

✓ + **4 pts** **Correct parametrization and domain**

+ **2 pts** Partial credits on parametrization

✓ + **4 pts** **Correct tangent and normal vector**

+ **2 pts** Partial credits on tangent and normal

✓ + **4 pts** **Correct double integral calculation**

+ **2 pts** Partial credits on double integral
- **1 pts** Almost there
+ **1 pts** Almost nothing correct
+ **0 pts** Nothing correct

QUESTION 4

4 Integration by parts 8 / 8

✓ + **1 pts** **Clear explanation**

✓ + **3 pts** **(a) correct**

✓ + **4 pts** **(b) correct**

+ **0 pts** Incorrect
+ **2 pts** (a) incomplete argument, but right idea
+ **2 pts** (b) incomplete argument, but right idea
+ **2 pts** (a) slight error
+ **3 pts** (b) slight error/unfinished
+ **1 pts** (a) started correctly, e.g. wrote integration by parts formula
+ **1 pts** (b) started correctly

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 2
2/22/2019

Name: NIKIL ROASHAN SELVAM
[redacted]
TA Section: 4F

Time Limit: 50 Minutes

Version ↑

This exam contains 12 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

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Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\dfrac{M_y}{M}, \dfrac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$

1. (6 points) Let $\mathcal{C}$ be the boundary of the rectangle $\mathcal{D} = \{-2 \le x \le 2\,,\ -1 \le y \le 1\}$ oriented counterclockwise and let

$$\mathbf{F}(x,y) = \left\langle e^{x^3-x} - 4y\,,\ \sin(e^y) + 3x^2y \right\rangle.$$

Evaluate the line integral

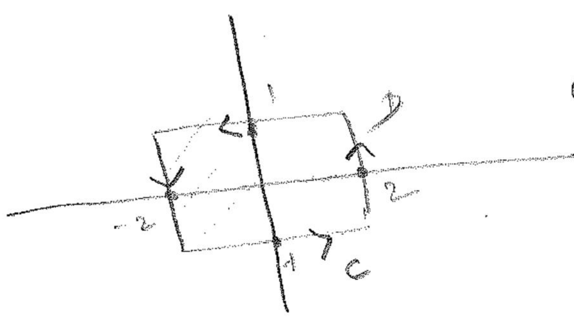
$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

Given $\vec{F}(x,y) = \langle e^{x^3-x} - 4y,\ \sin(e^y) + 3x^2y \rangle$.

$$\text{curl}_z \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= 6xy + 4$$

Given domain $D = \{-2 \le x \le 2,\ -1 \le y \le 1\}$.



C oriented counter clockwise

By Green's Theorem,

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_D \text{curl}_z \vec{F}\, dA$$

$$= \int_{-2}^{2} \int_{-1}^{1} (6xy + 4)\, dy\, dx$$

$$= \int_{-2}^{2} \left[3xy^2 + 4y \right]_{y=-1}^{y=1} dx$$

$$= \int_{-2}^{2} \left[3x + 4 - 3x - 4(-1) \right] dx$$

$$= \int_{-2}^{2} 8\,dx$$

$$= 8 \left[x \right]_{-2}^{2}$$

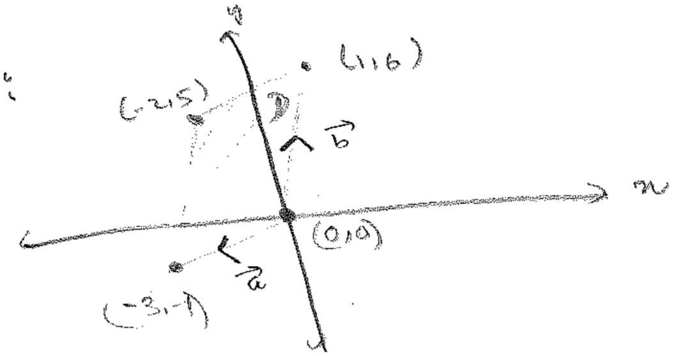
$$= 8 \cdot 4$$

$$= 32$$

$$\therefore \oint_C \vec{F} \cdot \vec{dx} = 32.$$

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-3,-1), (0,0), (1,6), (-2,5)$ (where distance is measured in meters) and with mass density $\delta(x,y) = \frac{1}{17}(3y-x)\ \mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$.

Domain D:



$\vec{a} = \langle -3, -1 \rangle \qquad \vec{b} = \langle 1, 6 \rangle$

Consider linear change of variables,

$(x,y) = u(-3,-1) + v(1,6)$

$= (-3u+v,\ -u+6v)$

$\therefore\ x = -3u+v \qquad y = -u+6v$

Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \left|\det\begin{bmatrix} -3 & 1 \\ -1, & 6 \end{bmatrix}\right| = |-18+1|$

$= +17$

$\therefore$ Given $\|^{gm}$ can be represented in u-v coordinates as $\{0 \le u \le 1,\ 0 \le v \le 1\}$.

Given, mass density of lamina $\delta = \frac{1}{17}(3y - x)$. Kg m^{-2}

$\therefore$ Total mass M

$$= \iint_D \delta(x,y)\, dA$$

$$= \iint_D \delta(x,y) \frac{\partial(x,y)}{\partial(u,v)}\, du\,dv$$

$$= \int_0^1\int_0^1 \frac{1}{17}\left(3[-u+6v] - [-3u+v]\right) \cdot (+17)\, du\,dv$$

$$= \int_0^1\int_0^1 +\left(-3u + 18v + 3u - v\right) du\,dv$$

$$= \int_0^1\int_0^1 +17v\, du\,dv$$

$$= \int_0^1 +17v\, [u]_0^1\, dv$$

$$= +17 \int_0^1 v\, dv$$

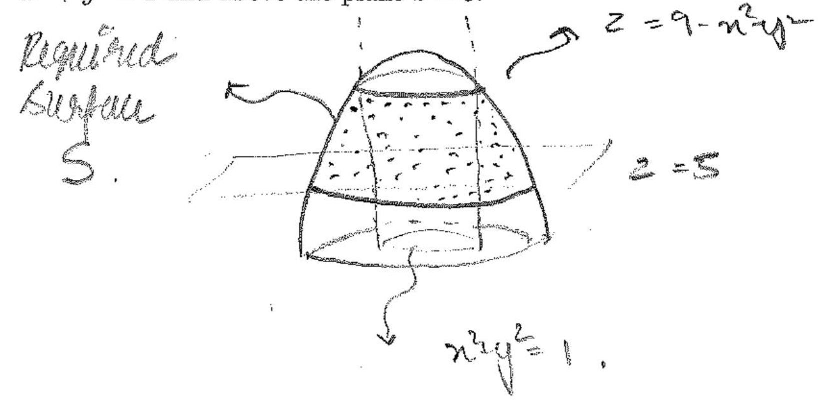
$$= +17 \left[\frac{v^2}{2}\right]_0^1$$

$$= \frac{17}{2}$$

$$= 8.5 \text{ Kg}$$

$\therefore$ M = 8.5 Kg

3. (12 points) Find the area of the part of the paraboloid $z = 9 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 5$.



Consider the following parametrization for S

$$G(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 9 - r^2 \rangle.$$

$$0 \leq \theta < 2\pi$$

For range of r:

Given S is outside cylinder $x^2+y^2=1$,

$$\therefore x^2+y^2 \geq 1$$
$$r^2\cos^2\theta + r^2\sin^2\theta \geq 1$$
$$\therefore r^2 \geq 1$$
$$\Rightarrow r \geq 1 \ [\because r \geq 0]$$

Also given S lies above plane $z = 5$

$$\therefore 9 - x^2 - y^2 \geq 5$$
$$9 - r^2 \geq 5$$
$$r^2 \leq 4$$
$$0 \leq r \leq 2 \ [\because r \geq 0]$$

Combining the two conditions, we get $1 \leq r \leq 2$

$$\frac{\partial G}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle \qquad \frac{\partial G}{\partial r} = \langle \cos\theta, \sin\theta, -2r \rangle$$

$$\left\| \frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} \right\| = \left\| \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix} \right\|$$

$$= \left\| \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle \right\|$$

$$= \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

Area of surface S

$$= \iint_S 1\, dS = \int_0^{2\pi}\int_1^2 \left\| \frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} \right\| dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\, dr\, d\theta$$

$$\left[\begin{array}{l|l} \text{let } 4r^2+1 = t^2 & 1 \le r \le 2 \\ \quad 8r\,dr = 2t\,dt & \sqrt{5} \le t \le \sqrt{17} \\ \quad r\,dr = \frac{1}{4}t\,dt & \end{array}\right.$$

$$\int_1^2 r\sqrt{4r^2+1}\,dr = \int_{\sqrt{5}}^{\sqrt{17}} t \cdot \frac{t\,dt}{4} = \frac{1}{4}\left[\frac{t^3}{3}\right]_{\sqrt{5}}^{\sqrt{17}} = \frac{1}{12}\left[17^{3/2} - 5^{3/2}\right] \Bigg]$$

$$= \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right)\int_0^{2\pi} d\theta$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \text{ sq. units } //$$

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve C oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

   (a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x, y)$, $g(x, y)$ we have the identity

$$\iint_{\mathcal{D}} f\Delta g\, dA - \iint_{\mathcal{D}} g\Delta f\, dA = \oint_C f\nabla g \cdot \mathbf{n}\, ds - \oint_C g\nabla f \cdot \mathbf{n}\, ds.$$

   (*Hint: Recall that* $\Delta f = \operatorname{div} \nabla f$)

As we know,

$$\iint_D f \operatorname{div} \vec{F} = \oint f\vec{F} \cdot \hat{n}\, ds - \iint_D \nabla f \cdot \vec{F}\, dA$$

Let $\vec{F} = \nabla g \Rightarrow \operatorname{div}\vec{F} = \operatorname{div} \nabla g = \Delta g$

$$\iint_D f\Delta g = \oint f\nabla g \cdot \hat{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA \rightarrow \text{Eq. } 1$$

Similarly, using integration by parts, we get

$$\iint_D g\Delta f = \oint g\nabla f \cdot \hat{n}\, ds - \iint_D \nabla g \cdot \nabla f\, dA \longrightarrow \text{Eq. } 2$$

Eq. 1 − Eq. 2

$$\Rightarrow \iint_D f\Delta g\, dA - \iint_D g\Delta f\, dA = \oint_C f\nabla g \cdot \hat{n}\, ds - \oint g\nabla f \cdot \hat{n}\, ds$$
$$- \iint_D \nabla f \cdot \nabla g\, dA + \iint_D \nabla f \cdot \nabla g\, dA$$

$[\because \nabla f \cdot \nabla g = \nabla g \cdot \nabla f]$.

$$\Rightarrow \iint_D f\Delta g\, dA - \iint_D g\Delta f\, dA = \oint_C f\nabla g \cdot \hat{n}\, ds - \oint g\nabla f \cdot \hat{n}\, ds$$

(b) Suppose that $f(x,y)$, $g(x,y)$ are smooth, non-zero, scalar functions satisfying the equations

$$\Delta f = \lambda f \quad \text{for all } (x,y) \in \mathcal{D},$$
$$\Delta g = \mu g \quad \text{for all } (x,y) \in \mathcal{D},$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x,y)$, $g(x,y)$ satisfy the boundary condition

$$f(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C},$$
$$g(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C}.$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x,y)g(x,y)\,dA = 0.$$

From part (a),

$$\iint_D f\Delta g\,dA - \iint_D g\Delta f\,dA = \oint_C f\nabla g\cdot\hat{n}\,ds - \oint_C g\nabla f\cdot\hat{n}\,ds$$

[Substituting $\Delta f = \lambda f$, $\Delta g = \mu g$ $\forall (x,y) \in D$, we get]

$$\mu\iint_D fg\,dA - \lambda\iint_D gf\,dA = 0 - 0$$

$\left[\begin{array}{l} \because f(x,y) = 0 \ \forall (x,y) \in C \\ \oint_C f\nabla g\cdot\hat{n}\,ds = 0 \end{array}\right]$ $\left[\begin{array}{l} \because g(x,y) = 0 \ \forall (x,y) \in C \\ \oint_C g\nabla f\cdot\hat{n}\,ds = 0 \end{array}\right]$

$$(\mu-\lambda)\iint_D fg\,dA = 0$$

i.e. $(\mu-\lambda)\iint_D f(x,y)g(x,y)\,dA = 0$

Whenever $\lambda \neq \mu$, $(\mu-\lambda) \neq 0$

$$\Rightarrow \iint_D f(x,y)\,g(x,y)\,dA = 0$$

Hence, proved. //